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Nonlinear excitation of long-trapped waves by a group of short swells

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A nonlinear theory is presented for the resonance of long gravity waves trapped on an uneven bottom when a long packet of short swells is incident. By allowing the trapped wave to be comparable in amplitude to the incident swells, the transient evolution of trapped waves is studied from initial growth through maturity to final decay, for swell packets of finite duration. For a totally submerged ridge, it is found that, while the trapped waves are resonated by second-order periodic modulations of the swell envelope, energy transfer to short swells and nonlinear emission of long waves act as damping mechanisms to limit the amplification. Bottom friction is not qualitatively crucial and affects the results only quantitatively. For a closed beach, breaking of short swells is dealt with empirically but resonant modulation is still the primary factor in exciting surf beats.

1. Introduction

Trapped waves can occur on a sloping beach (Stokes 1846; Ursell 1952), along a coastline with abrupt depth changes (Longuet-Higgins 1967) or over a submarine ridge (Munk, Snodgrass & Carrier 1956; Buchwald 1969). According to the linearized theory, these waves may be excited by transient incident waves, or by direct atmospheric forcing (Greenspan 1956) but not by steady sinusoidal incident waves. Field measurements made by Munk (1949) and Tucker (1950) have shown that long-period oscillations exist on natural beaches in the period range of 1–5 min. In particular the envelope of the long-period waves is found to resemble closely the swell envelope. For storms of *c.* 1–2 day duration the long wave height is roughly one-tenth of the swell amplitude. Munk attributed this correlation to nonlinear interaction between long and short waves and termed the phenomenon *surf beat*. Gallagher (1971) has given a theory in which groups of swells with a narrow frequency band generate long-period forcing at the second order and hence resonate a long-period edge wave. In his theory, however, rather drastic simplifications were made with regard to the shoaling swells. In addition, bottom friction was needed to render the resonant amplitude finite. A related theory has been put forth by King & Smith (1978) who also neglected the breaking zone, and found bottom friction to be crucial in the initial stage of side-band instability. In these theories the long waves are of second order in swell slope.

As demonstrated by Guza & Bowen (1976) and Minzoni & Whitham (1977), edge waves may also be excited by a nonlinear subharmonic mechanism. Since the most energetic swells have the period range of $8 \sim 15$ s, edge waves so excited can only have twice the period of $16 \sim 30$ s, far below that of the surf beats. This mechanism is

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therefore of greater relevance to small-scale motion and to the formation of small beach cusps. Of interest in their theory is that, instead of friction, damping due to nonlinear radiation of second harmonic waves plays an important role in the resonance process.

In this paper we re-examine the mechanism of nonlinear resonance by slowly modulated swells. In order to avoid empirical uncertainties, an inviscid theory is first worked out for the excitation of long wave trapped on a submarine ridge of small slope. The incident short swells pass over the ridge without significant reflection or breaking. In contrast to earlier theories, we allow the long wave to grow to magnitudes comparable to the incident swells. It is found that the nonlinear radiation of second harmonics of long waves produces a damping effect which renders the resonant growth finite. We also find that the interaction of short and long waves transfers energy from the latter to the former. The effect of bottom friction is added later merely as a correction which is seen to enhance damping but does not play a critical role.

The same analysis is then modified for surf beats on a beach. Since the short swells break in the surf zone where potential theory is no longer valid, empirical relations customary in long-shore current theory are invoked. On the other hand, the long waves are assumed to be only indirectly affected by the breaking swells through nonlinear interaction. It is then found that the breaking waves in the surf zone transfer energy from swells to the long waves, causing instability. However, sample calculations indicate that the unstable growth rate is weak and can be easily overcome by radiation and even more easily by bottom friction, so that resonant forcing is still the most important cause for surf beats.

In contrast to the sub-harmonic theory of Minzoni & Whitham (1977) and Rockliff (1978) where two time scales exist, our theory involves four time scales and two space scales. Furthermore, in order for the trapped wave to be in the 1–5 min range, the bottom slope cannot be much milder than the swell steepness. Consequently, the required WKB analysis† is very lengthy here. Moreover, to determine the eigenmode for a smooth submerged ridge, a numerical technique is needed. While the essential steps of our analysis are described, most of the manipulations are omitted for brevity.

2. General plan of solution

In terms of the velocity potential Φ and the free surface displacement ζ, the governing equations are

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\Phi = 0, \quad -h < z < \zeta, \tag{2.1}$$

$$\frac{\partial\Phi}{\partial z} = -\frac{\partial\Phi}{\partial x}\frac{\partial h}{\partial x}, \quad z = -h(x) \tag{2.2}$$

$$\left.\begin{aligned} &\frac{\partial\zeta}{\partial t}+\frac{\partial\Phi}{\partial x}\frac{\partial\zeta}{\partial x}+\frac{\partial\Phi}{\partial y}\frac{\partial\zeta}{\partial y} = \frac{\partial\Phi}{\partial z}, &(2.3)\\ &g\zeta+\frac{\partial\Phi}{\partial t}+\frac{1}{2}\left[\left(\frac{\partial\Phi}{\partial x}\right)^2+\left(\frac{\partial\Phi}{\partial y}\right)^2+\left(\frac{\partial\Phi}{\partial z}\right)^2\right] = 0, &(2.4)\end{aligned}\right\} \quad z = \zeta.$$

† The method has been used before by Chu & Mei (1970) for Stokes waves on a not-so-mild beach.

Let ϵ be the characteristic slope of the sea bottom, and ϵ^{-1} times the swell wavelength be the characteristic length of both the trapped long wave and the swell envelope. Hence ϵ is the measure of slow modulation. Since it is well known that the most physically interesting phenomena occur when the slow modulation of the leading-order terms is comparable to nonlinear terms, we assume the steepness of the short swells to be $O(\epsilon)$ also. It is convenient to introduce the slow variables

$$\left.\begin{aligned} &y_1 = \epsilon y, \\ &x_1 = \epsilon x, \quad x_2 = \epsilon^2 x, \quad x_3 = \epsilon^3 x, \quad \ldots, \\ &t_1 = \epsilon t, \quad t_2 = \epsilon^2 t, \quad t_3 = \epsilon^3 t, \quad \ldots. \end{aligned}\right\} \tag{2.5}$$

In particular, we shall confine our attention to periodic variations on the y_1 scale only. Hence $y, y_2, y_3, \ldots$ are not expected to arise. The swell envelope is in general a function of $y_1, x_1, x_2, x_3, \ldots, t_1, t_2, t_3, \ldots$. On the other hand, since the trapped wave is expected to diminish exponentially for $|x_1| \gg 1$, there is no need to allow its amplitude to depend on $x_2, x_3, \ldots$, as long as attention is confined to the region of $x_1 = O(1)$. To have some quantitative idea, we let the bottom slope be $\epsilon = 0{\cdot}05$ and the incident swells have the period $2\pi/\omega = 10$ s. Then $t_1 = O(3{\cdot}3 \text{ min})$ which is a typical period of surf beats, $t_2 = O(1 \text{ hr})$ which is the typical period of storm surges, and $t_3 = O(20 \text{ hrs})$ which is the typical duration of a storm. Tides, which have the time scale of t_3, are ignored.

Since the problem is weakly nonlinear almost everywhere, it is convenient to express the free surface conditions in terms of Φ alone and to expand about $z = 0$:

$$\begin{aligned} \Phi_{tt} + g\Phi_z = {} & [-\tfrac{1}{2}(\nabla\Phi)^2 + \tfrac{1}{2}\Phi_t\Phi_{zt}]_t - [\Phi_x\Phi_t]_x - [\Phi_y\Phi_t]_y \\ & + \left[\frac{1}{2g}(\Phi_t(\nabla\Phi)^2)_z - \frac{1}{g^2}\Phi_t\Phi_{zt}^2 - \frac{1}{2g^2}\Phi_{zzt}\Phi_t^2\right]_t \\ & + \left[-\tfrac{1}{2}\Phi_x(\nabla\Phi)^2 + \frac{1}{g}\Phi_x\Phi_t\Phi_{zt} + \frac{1}{2g}\Phi_t^2\Phi_{xz}\right]_x \\ & + \left[-\tfrac{1}{2}\Phi_y(\nabla\Phi)^2 + \frac{1}{g}\Phi_y\Phi_t\Phi_{zt} + \frac{1}{2g}\Phi_t^2\Phi_{yz}\right]_y + O(\Phi^4) \quad \text{at} \quad z = 0. \end{aligned} \tag{2.6}$$

The following expansions will then be substituted:

$$\left.\begin{aligned} \Phi &= \sum_{n=0}^{\infty} \epsilon^n \sum_{m=-n}^{n} \phi_{nm} e^{im\psi/\epsilon} \\ \zeta &= \sum_{n=1}^{\infty} \epsilon^n \sum_{m=-n}^{n} \eta_{nm} e^{im\psi/\epsilon} \end{aligned}\right\} \quad \text{with} \quad \left\{\begin{aligned} &\phi_{nm} = \phi^*_{n,-m}, &(2.7a) \\ &\eta_{nm} = \eta^*_{n,-m}, &(2.7b) \end{aligned}\right.$$

where $*$ denotes complex conjugate. The functions ψ, and η_{nm}, ϕ_{nm} depend on the slow variables and ϕ_{nm} further on z. We also denote

$$\mathbf{k} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}\right)\psi, \quad \omega = -\frac{\partial\psi}{\partial t_1}. \tag{2.8}$$

It is important that the expansion begins with ϕ_{00} at $O(\epsilon^0)$, which implies that the long-period trapped wave ($\epsilon\eta_{10}$) can be as large as the swells. Now equation (2.2) becomes

$$\frac{\partial\Phi}{\partial z} = -\epsilon\frac{\partial h}{\partial x_1}\frac{\partial\Phi}{\partial x} \quad \text{at} \quad z = -h(x_1). \tag{2.9}$$

Furthermore, we let the first-order swells be normally incident and the frequency ω be constant:

$$\mathbf{k} = (k(x_1), 0), \quad \omega = \text{constant}. \tag{2.10}$$

Higher-order modulations will be incorporated in the amplitudes. Substituting (2.7) into (2.1), (2.6), and (2.9) and separating terms of different orders and harmonics, we obtain the governing conditions for each set of (n, m)

$$\left(\frac{\partial^2}{\partial z^2} - m^2k^2\right)\phi_{nm} = F_{nm}, \quad -h < z < 0, \tag{2.11a}$$

$$\left(g\frac{\partial}{\partial z} - m^2\omega^2\right)\phi_{nm} = G_{nm}, \quad z = 0, \tag{2.11b}$$

$$\frac{\partial}{\partial z}\phi_{nm} = H_{nm}, \quad z = -h. \tag{2.11c}$$

The inhomogeneous terms F_{nm}, G_{nm}, H_{nm} are obtained in the manner of Chu & Mei (1970) but include many new terms because of ϕ_{00} and the multiple scales; they will be given when needed.

For the zeroth and first harmonics ($m = 0$ and 1) the boundary-value problems (2.11) have non-trivial homogeneous solutions. Let ϕ^h denote the corresponding homogeneous solution in each case; the inhomogeneous problems at higher orders must be subjected to the following solvability constraint

$$\int_{-h}^{0}\left(\phi^h\frac{\partial^2\phi_{nm}}{\partial z^2} - \phi_{nm}\frac{\partial^2\phi^h}{\partial z^2}\right)dz = \left[\phi^h\frac{\partial\phi_{nm}}{\partial z} - \phi_{nm}\frac{\partial\phi^h}{\partial z}\right]_{z=-h}^{z=0}, \tag{2.12}$$

which will be referred to as VSC (for Vertical Solvability Condition). It may be shown that for the zeroth harmonic ($m = 0$) $\phi^h = 1$ and

$$H_{n0} = -\frac{\partial h}{\partial x_1}\frac{\partial}{\partial x_1}\phi_{n-2,0} + \sum_{i=1}^{n-2} -\frac{\partial h}{\partial x_1}\frac{\partial}{\partial x_{i+1}}\phi_{n-2-i,0}, \tag{2.13a}$$

$$F_{n0} = -\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2}\right)\phi_{n-2,0} + \sum_{i=1}^{n-2}\sum_{j=1}^{i+1} -\frac{\partial^2}{\partial x_j\partial x_{2+i-j}}\phi_{n-2-i,0}, \tag{2.13b}$$

and G_{n0} depends on potentials of order up to $n-2$ in a more complex manner. Note that the summations disappear for $n = 1$ and 2. Equation (2.12) may then be reduced to

$$\nabla_1\cdot\int_{-h}^{0}\nabla_1\phi_{n-2,0}\,dz = -\frac{1}{g}G_{n0} + \int_{-h}^{0}\hat{F}_{n0}\,dz + \hat{H}_{n0}, \quad \nabla_1 \equiv \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}\right), \tag{2.14}$$

where $\hat{F}_{n0}$ and $\hat{H}_{n0}$ are the summation terms in (2.13a, b) respectively. Thus the solvability of ϕ_{n0} imposes a constraint on $\phi_{n-2,0}$. For the first harmonic ($m = 1$), $\phi^h = \cosh k(z+h)$; H_{n1}, F_{n1} and G_{n1} depend on potentials of order up to $n-1$. Equation (2.12) then becomes

$$H_{n1} + \int_{-h}^{0}F_{n1}\cosh k(z+h)\,dz = \frac{1}{h}\cosh kh G_{n1}, \tag{2.15}$$

which imposes a constraint on $\phi_{n-1,1}$. Subsequent details are rather tedious, and it is useful to sketch first the key steps to be followed.

At $O(\epsilon^0)$, ϕ_{00} is seen to depend on the slow horizontal co-ordinates and time $(x_1, y_1, t_1, \ldots)$, and hence represents long waves. At $O(\epsilon)$, ϕ_{11} corresponds to the short swell with unknown amplitude $A(x_1, y_1, t_1, \ldots)$. At $O(\epsilon^2)$, vertical solvability of ϕ_{20} gives a homogeneous long-wave equation for ϕ_{00} which can be solved as a free trapped wave of unknown amplitude $D(t_1, t_2, \ldots)$. Application of VSC (2.15) to ϕ_{21} determines A in terms of its boundary value at $x_1 \sim -\infty$, i.e. beyond the range of the trapped waves. ϕ_{21} may now be solved with a new homogeneous solution of unknown amplitude B which includes the modulational disturbance that forces resonance. Similar application of VSC (2.15) to ϕ_{31} determines B in terms of its known value at $x_1 \sim -\infty$. On the other hand VSC (2.14) applied to ϕ_{30} yields for ϕ_{10} an inhomogeneous long-wave equation which has ϕ_{00} as its homogeneous solution. Thus a horizontal solvability condition (HSC) exists which gives a constraint for the amplitude of ϕ_{00}, i.e. D with regard to t_2. The constraint turns out to be trivial. Finally, similar application of VSC (2.14) to ϕ_{40}, followed by imposing HSC to the resulting long-wave equation for ϕ_{20}, yields the evolution equation for D with respect to t_3 which may be solved for given initial data and the boundary values A and B of the incident swells. It is the details of the fourth-order term ϕ_{40} which require daunting algebra.

3. Perturbation analysis of the interaction problem

3.1. *The long trapped wave ϕ_{00}*

For the long-scaled motion ($m = 0$) we have

$$F_{00} = F_{10} = 0, \quad F_{20} = -\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2}\right)\phi_{00}, \tag{3.1a}$$

$$G_{00} = G_{10} = 0, \quad G_{20} = -\frac{\partial^2 \phi_{00}}{\partial t_1^2}, \tag{3.1b}$$

$$H_{00} = H_{10} = 0, \quad H_{20} = -\frac{\partial h}{\partial x_1}\frac{\partial \phi_{00}}{\partial x_1}. \tag{3.1c}$$

Consequently ϕ_{00} and ϕ_{10} are independent of z but otherwise indeterminate functions of x_1, y_1, t_1, t_2, ... The potential ϕ_{20} may be formally solved:

$$\phi_{20} = \tfrac{1}{2}F_{20}z^2 - \frac{1}{g}\frac{\partial^2 \phi_{00}}{\partial t_1^2}z + N, \tag{3.2}$$

where N is an indeterminate function of the slow variables. Applying VSC (2.14) to ϕ_{20} we get the long-wave equation for ϕ_{00}

$$\frac{\partial^2 \phi_{00}}{\partial t_1^2} - g\nabla_1 . (h\nabla_1 \phi_{00}) = 0, \tag{3.3}$$

which is homogeneous. From Bernoulli's equation, the corresponding free surface height is $\epsilon\eta_{10}$, where

$$\eta_{10} = -\frac{1}{g}\frac{\partial \phi_{00}}{\partial t_1}. \tag{3.4}$$

A homogeneous solution is

$$\phi_{00} = -\frac{igD}{2\Omega} L_\nu(x_1)\cos Ky_1 e^{-i\Omega t_1} + *, \tag{3.5}$$

where $D(t_2, t_3, \ldots)$ is the amplitude of η_{10}, and L_ν satisfies

$$g(hL_\nu')' + (\Omega^2 - ghK^2) L_\nu = 0. \tag{3.6}$$

For a trapped wave we insist that

$$L_\nu \downarrow 0 \quad \text{as} \quad |x_1| \sim \infty. \tag{3.7}$$

For this to be true it is necessary that

$$\Omega^2 - ghK^2 < 0 \quad \text{as} \quad |x_1| \sim \infty. \tag{3.8}$$

In addition, L_ν must also be bounded everywhere. Thus (3.5)–(3.7) define an eigenvalue problem where L_ν is the νth eigenfunction and K the eigenvalue, Ω being regarded as given. For a general $h(x_1)$ the eigenvalue problem can be solved numerically. A convenient and numerical method employing finite elements will be described in appendix A. For the special case of a plane beach $h = -x_1$, $x_1 < 0$ the analytical solution is

$$L_\nu = e^{Kx_1}\mathscr{L}_\nu(-2Kx_1) \tag{3.9a}$$

with $\mathscr{L}_\nu$ = Laguerre polynomial, corresponding to edge waves, where

$$\Omega^2 = gK(2\nu+1); \quad \nu = 0, 1, 2, \ldots. \tag{3.9b}$$

In particular for the lowest mode $\nu = 0$, $\mathscr{L}_0 = 1$, and $\Omega^2 = gK$. Although they have the physical dimensions of frequency and wavenumber, Ω and K are distorted because of the use of slow variables x_1 and t_1.

3.2. *The short swells ϕ_{11}*

For $n = 1$, $m = 1$, $F_{11} = H_{11} = G_{11} = 0$, the solution is simply

$$\phi_{11} = -\frac{igA}{2\omega}\frac{\cosh Q}{\cosh q} \quad \text{with} \quad Q \equiv k(z+h), \quad q \equiv kh, \tag{3.10}$$

where

$$\omega^2 = gk \tanh q. \tag{3.11}$$

Thus $A(x_1, y_1, t_1, \ldots)$ is the amplitude of the incident swells.

For $(n = 2, m = 1)$ we have

$$F_{21} = -i\frac{\partial k\phi_{11}}{\partial x_1} - ik\frac{\partial \phi_{11}}{\partial x_1}, \tag{3.12a}$$

$$H_{21} = -\frac{\partial h}{\partial x_1} ik\phi_{11}\bigg|_{z=-h}, \tag{3.12b}$$

$$G_{21} = i2\omega\frac{\partial \phi_{11}}{\partial t_1}\bigg|_{z=0} + ig\left[k\frac{\partial \phi_{00}}{\partial x_1} - \frac{k^2}{2\omega\cosh^2 q}\frac{\partial \phi_{00}}{\partial t_1}\right] A; \tag{3.12c}$$

these may be substituted into VSC (2.15) to give

$$-\frac{\partial h}{\partial x_1} ik\phi_{11}\bigg|_{z=-h} - \int_{-h}^{0} dz \cosh Q\left[\frac{\partial}{\partial x_1}(ik\phi_{11}) + ik\frac{\partial \phi_{11}}{\partial x_1}\right] = \frac{1}{g}\cosh q G_{21}. \tag{3.13}$$

Multiplying (3.13) by $gA^*/2\omega \cosh q$, then adding the resulting equation to its own complex conjugate, we obtain the usual law of conservation of wave action

$$\frac{\partial}{\partial t_1}\left(\frac{|A|^2}{\omega}\right)+\frac{\partial}{\partial x_1}\left(C_g\frac{|A|^2}{\omega}\right) = 0, \tag{3.14}$$

where C_g is the group velocity of infinitesimal swells,

$$C_g = \frac{\omega}{2k}(1+2q/\sinh 2q). \tag{3.15}$$

Because of (2.10) and the fact that (3.14) is independent of the long waves ϕ_{00}, we may assume that $|A|^2$ is independent of t_1. Consequently, (3.14) may be integrated to give the usual shoaling formula,

$$|A| = (C_g^\circ/C_g)^{\frac{1}{2}}|A^\circ|, \tag{3.16}$$

where the superscript $(\;)^\circ$ refers to $x_1 \sim -\infty$, i.e. beyond the region of the trapped waves. In the absence of ϕ_{00}, A° would be real. Now the presence of ϕ_{00} in G_{21} means that the phase of A is affected by the long waves. Multiplying (3.13) by $gA/2\omega \cosh q$ and using (3.15), we get

$$\frac{\partial}{\partial t_1}A^2+\frac{\partial}{\partial x_1}(C_gA^2) = -2i\left[k\frac{\partial \phi_{00}}{\partial x_1}-\frac{k^2}{2\omega\cosh^2 q}\frac{\partial\phi_{00}}{\partial t_1}\right]A^2. \tag{3.17}$$

Let the phase of A be θ, i.e.

$$A = |A|e^{i\theta}; \tag{3.18}$$

it follows from (3.17) that

$$\left(\frac{\partial}{\partial t_1}+C_g\frac{\partial}{\partial x_1}\right)\theta = -\left[k\frac{\partial}{\partial x_1}-\frac{k^2}{2\omega\cosh^2 q}\frac{\partial}{\partial t_1}\right]\left[-\frac{igD}{2\Omega}L_\nu \cos Ky_1 e^{-i\Omega t_1}+*\right]. \tag{3.19}$$

Assume θ to be of the form

$$\theta = \frac{-igD}{2\Omega}\Theta(x_1)\cos Ky_1 e^{-i\Omega t_1}+*; \tag{3.20}$$

then

$$\frac{\partial\Theta}{\partial x_1}-\frac{i\Omega}{C_g}\Theta = -\frac{1}{C_g}\left(kL_\nu'+\frac{i\Omega k^2}{2\omega\cosh^2 q}L_\nu\right), \tag{3.21}$$

where primes denote derivatives with respect to x_1. Equation (3.21) may be easily solved:

$$\Theta = -e^{iX}\int_{-\infty}^{x_1}\frac{dx_1}{C_g}e^{-iX(x_1)}\left(kL_\nu'+\frac{i\Omega k^2}{2\omega\cosh^2 q}L_\nu\right) \tag{3.22}$$

with

$$X(x_1) = \Omega\int_{-\infty}^{x_1}\frac{dx_1}{C_g}.$$

Thus ϕ_{11} is completely solved.

Note that as $x_1 \sim -\infty$, i.e. outside the range of the trapped wave ϕ_{00}, $\theta \downarrow 0$ and $A \to A^\circ$. Thus A° must be a prescribed function of $(x_2, x_3, t_2, t_3, \ldots)$ and specifies the envelope of the incident swells, to the first order.

3.3. *The potential* ϕ_{21}

Having satisfied VSC (2.15), solution for ϕ_{21} is assured. The formal solution is the same as that in Chu & Mei (1970):

$$\phi_{21} = -\underbrace{\frac{igB\cosh Q}{2\omega\cosh q}}_{\phi_{21}^h} - \underbrace{\frac{gA}{2\omega\cosh q}(\alpha_1 Q\cosh Q + \alpha_2 Q\sinh Q + \alpha_3 Q^2\cosh Q)}_{\phi_{21}^p}, \quad (3.23)$$

where

$$\alpha_1 = h', \quad \alpha_2 = [(A/2\omega\cosh q)^2]'/2k(A/2\omega\cosh q)^2, \quad \alpha_3 = k'/2k^2,$$

and A is given by (3.18). All except the first term on the right of (3.23) is the particular solution and will be denoted by ϕ_{21}^p. The amplitude B of the homogeneous solution ϕ_{21}^h must now be determined to provide information needed for ϕ_{40}. For the resonance of the trapped wave, we expect that at the second order the incident swell envelope B contains a part which is in harmony with ϕ_{00}. The associated term will be called the *resonant modulation*. The constraint for B is found by applying VSC (2.15) to ϕ_{31}, for which the following results are needed:

$$F_{31} = -\left[\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2}\right)\phi_{11} + i\frac{\partial}{\partial x_1}(k\phi_{21}) + ik\frac{\partial\phi_{21}}{\partial x_1} + 2ik\frac{\partial\phi_{11}}{\partial x_2}\right], \quad (3.24a)$$

$$H_{31} = -\frac{\partial h}{\partial x_1}\left(ik\phi_{21} + \frac{\partial\phi_{11}}{\partial x_1}\right)\Big|_{z=-h}, \quad (3.24b)$$

$$G_{31} = \{i2\omega[(\phi_{21})_{t_1} + (\phi_{11})_{t_2}] - (\phi_{11})_{t_1t_1}\}_{z=0} + (\phi_{21},\phi_{00}) + (\phi_{00},\phi_{11}) + (\phi_{10},\phi_{11}) + (\phi_{11},\phi_{11},\phi_{11}). \quad (3.24c)$$

The terms (,) represent symbolically nonlinear contributions of various couplings. After lengthy algebra, the result may be summarized as follows:

$$G_{31} = i2\omega\frac{\partial\phi_{21}^h}{\partial t_1} + ig\left(k\frac{\partial\phi_{00}}{\partial x_1} - \frac{k^2}{2\omega\cosh^2 q}\frac{\partial\phi_{00}}{\partial t_1}\right)B + \hat{G}_{31}, \quad z = 0, \quad (3.24d)$$

where $\hat{G}_{31}$ depends on A and D,

$$\begin{aligned}\hat{G}_{31} = {} & iA\omega g\left[\frac{k^2|A|^2}{16}(9\coth^4 q - 10\coth^2 q + 9)\right] + ig\left(k\frac{\partial\phi_{10}}{\partial x_1} - \frac{k^2}{2\omega\cosh^2 q}\frac{\partial\phi_{10}}{\partial t_1}\right)A \\ & + 2i\omega(\phi_{21}^p)_{t_1} + 2i\omega(\phi_{11})_{t_2} - (\phi_{11})_{t_1t_1} - 2k\omega\phi_{21}^p(\phi_{00})_{x_1} + (\phi_{00})_{t_1}\left[\frac{-\omega^2}{g}(\phi_{21}^p)_z + k^2\phi_{21}^p\right] \\ & - [ik\phi_{11}(\phi_{00})_{x_1}]_{t_1} + i\omega(\phi_{11})_{x_1}(\phi_{00})_{x_1} + i\omega(\phi_{00})_{y_1}(\phi_{11})_{y_1} - \frac{1}{g}[i\omega(\phi_{11})_z(\phi_{00})_{t_1}]_{t_1} \\ & - \frac{\omega^2}{g}(\phi_{11})_z(\phi_{00})_{t_2} - \frac{i\omega}{g}(\phi_{00})_{t_1}(\phi_{11})_{zt_1} + k^2\phi_{11}(\phi_{00})_{t_2} - ik(\phi_{11})_{t_1}(\phi_{00})_{x_1} - ik(\phi_{00})_{t_1}(\phi_{11})_{x_1} \\ & - [(-i\omega\phi_{11}(\phi_{00})_{x_1} + ik\phi_{11}(\phi_{00})_{t_1}]_{x_1} + [i\omega\phi_{11}(\phi_{00})_{y_1}]_{y_1} - \frac{i\omega}{2g}[-i\omega(\phi_{11})_z(\nabla_1\phi_{00})^2 \\ & + 2ik(\phi_{11})_z(\phi_{00})_{t_1}(\phi_{00})_{x_1}] + \frac{\omega^2}{2g^2}(\phi_{11})_{zz}[(\phi_{00})_{t_1}]^2 + \frac{k^2}{2}\phi_{11}(\nabla_1\phi_{00})^2 + k^2\phi_{11}[(\phi_{00})_{x_1}]^2 \\ & + \frac{k\omega}{g}(\phi_{11})_z(\phi_{00})_{x_1}(\phi_{00})_{t_1} - \frac{k^2}{2g}(\phi_{11})_z[(\phi_{00})_{t_1}]^2, \quad z = 0. \end{aligned} \quad (3.24e)$$

Invoking VSC (2.15) on the boundary-value problem for ϕ_{31}, we then have

$$-h_{x_1}[ik\phi_{21}+(\phi_{11})_{x_1}]|_{z=-h}+\int_{-h}^{0}[-\nabla_1^2\phi_{11}-i(k\phi_{21})_{x_1}-ik(\phi_{21})_{x_1}-2ik(\phi_{11})_{x_2}]\cosh Q\,dz = \frac{1}{g}\cosh q\, G_{31}. \quad (3.25)$$

Using the explicit expressions of ϕ_{11} and ϕ_{21}, and multiplying (3.25) by $gB/2\omega\cosh q$, we obtain after a good deal of algebra that

$$\left\{\frac{\partial B^2}{\partial t_1}+\frac{\partial}{\partial x_1}(C_g B^2)+2i\left(k\frac{\partial \phi_{00}}{\partial x_1}-\frac{k^2}{2\omega\cosh^2 q}\frac{\partial\phi_{00}}{\partial t_1}\right)B^2\right\}+2BC_g\frac{\partial A}{\partial x_2} = 2B\left[\frac{S}{\cosh q}-\frac{1}{g}\hat{G}_{31}\right], \quad (3.26)$$

where

$$S = -\frac{\partial h}{\partial x_1}\left(ik\phi_{21}^p+\frac{\partial\phi_{11}}{\partial x_1}\right)_{-h}+\int_{-h}^{0}dz\cosh Q\left[-\nabla_1^2\phi_{11}-i\frac{\partial}{\partial x_1}(k\phi_{21}^p)-ik\frac{\partial}{\partial x_1}(\phi_{21}^p)\right]. \quad (3.27)$$

The explicit form of S is a slight modification of a similar term in Chu & Mei (1970) and is very lengthy. For convenience it is quoted in appendix B. Since part of (3.26) resembles (3.17) we try

$$B = (C_g^\circ/C_g)^{\frac{1}{2}}\, b e^{i\theta}; \quad b = b^r + ib^i, \quad (3.28)$$

where θ is given by (3.20) and (3.22) and where b^r and b^i are respectively the real and imaginary parts of b. Substituting this into (3.26) and using (3.19), we get a linear equation for b:

$$\frac{\partial b}{\partial t_1}+C_g\frac{\partial b}{\partial x_1} = -\left(\frac{C_g}{C_g^\circ}\right)^{\frac{1}{2}}\left(C_g\frac{\partial A}{\partial x_2}+\frac{1}{g}\hat{G}_{21}-\frac{1}{\cosh q}S\right)e^{-i\theta}. \quad (3.29)$$

Note that ϕ_{21}, which is from the second order, couples with ϕ_{11} through quadratic terms in (2.6) to give zeroth harmonics with respect to the swell frequency ω. Furthermore these zeroth harmonics have only derivatives with respect to slow variables; hence they contribute at the *fourth order* to G_{40}. Since such contributions must come from the combination

$$\phi_{21}^*\phi_{11}+\phi_{21}\phi_{11}^*,$$

only the part of ϕ_{21} which is in phase with ϕ_{11} matters. In view of the definition (3.28), only the real part b^r in the homogeneous solution ϕ_{21}^h matters to G_{40}. It is therefore only necessary to consider the real part of (3.29), which may be rewritten as

$$\left(\frac{\partial}{\partial t_1}+C_g\frac{\partial}{\partial x_1}\right)b^r = \left(\frac{\partial A^\circ}{\partial t_2}+C_g^\circ\frac{\partial A^\circ}{\partial x_2}\right)+\left[\frac{-igBA^\circ}{2\Omega}W(x_1)\cos Ky_1 e^{-i\Omega t_1}+*\right], \quad (3.30)$$

where

$$W = \left\{\frac{|A|}{A^\circ}\left(\frac{C_g}{C_g^\circ}\right)^{\frac{1}{2}}\right\}\left\{\frac{\omega}{k}\Theta'\left[\frac{3k'}{4k^2}\left(q^2-2q\coth 2q+1+\frac{2q^3}{3\sinh 2q}\right)\right.\right.$$
$$\left.-\frac{h'}{2}\left(2q+\coth 2q+\frac{1+2q}{\cosh 2q}\right)\right]-\frac{\omega}{k^2}\left[\theta''+2\theta'\left(\ln\frac{|A|}{\cosh q}\right)'\right]$$
$$\times\left[\frac{q}{2\sinh 2q}+\frac{q}{2}\coth 2q\right]+\frac{\omega K^2}{2k^2}\left(\frac{1}{2}+\frac{q}{\sinh 2q}\right)\Theta-\frac{\Omega^2}{2\omega}\Theta$$

$$+ i\Omega\Theta(\alpha_1 q + \tilde{\alpha}_2 q \tanh q + \alpha_3 q^2) + \frac{i\Omega}{k} q \tanh q \Theta'$$

$$+ \frac{1}{2\omega}(\alpha_1 q + \tilde{\alpha}_2 q \tanh q + \alpha_3 q^2)(-2k\omega L_\nu' - i\Omega k^2 L_\nu)$$

$$+ \frac{k}{2\omega}[\alpha_1(1 + q\tanh q) + \tilde{\alpha}_2(\tanh q + q) + \alpha_3(2q + q^2 \tanh q)]\left(\frac{i\omega^2\Omega}{g} L_\nu\right)$$

$$- \frac{i\Omega k}{2\omega} L_\nu' - \frac{\omega^2\Omega^2}{2g^2} L_\nu + \tfrac{1}{2}K^2 L_\nu - \frac{1}{|A|}\left[\tfrac{1}{2}|A|' L_\nu' + \frac{i\Omega k}{2\omega}|A|' L_\nu + \left(\frac{|A|}{2} L_\nu' + \frac{i\Omega k}{2\omega}|A| L_\nu\right)'\right]\Bigg\} \tag{3.31}$$

and where

$$\tilde{\alpha}_2 = \operatorname{Re}\alpha_2 = \frac{\cosh^2 q}{2k|A|^2}\left[\frac{|A|^2}{\cosh^2 q}\right]_{x_1}.$$

Now, since A° is not a function of x_1 or t_1, we must insist for the boundedness of b^r that

$$\frac{\partial A^\circ}{\partial t_2} + C_g \frac{\partial A^\circ}{\partial x_2} = 0. \tag{3.32}$$

The solution for b^r must then be of the form

$$b^r = \hat{b}(x_1)\cos K y_1 e^{-i\Omega t_1} + *, \tag{3.33}$$

where $\hat{b}$ is complex. It is evident that $\hat{b}$ is governed by

$$\left(\frac{\partial}{\partial x_1} - \frac{i\Omega}{C_g}\right)\hat{b} = -\frac{igDA^\circ}{2\Omega}\frac{W}{C_g}. \tag{3.34}$$

As the boundary condition for (3.34), we require that the amplitude B of ϕ_{21}^h at $x_1 \sim -\infty$ is given by

$$B \to b^\circ \cos K y_1 \exp\left[i\Omega\left(\frac{x_1}{C_g^\circ} - t_1\right)\right], \quad x_1 \sim -\infty, \tag{3.35}$$

or

$$\hat{b} \to \tfrac{1}{2} b^\circ \exp(i\Omega x_1 / C_g^\circ)$$

with $b^\circ(t_2, t_3, \ldots)$ being the amplitude of the incident first harmonic in Ω. The solution of (3.34) subject to the boundary condition (3.35) is simply

$$\hat{b} = \tfrac{1}{2} b^\circ e^{iX} - \frac{igDA^\circ}{2\Omega} e^{iX} \int_{-\infty}^{x_1} e^{-iX(\xi)} \frac{W}{C_g} d\xi. \tag{3.36}$$

Combining (3.36) with (3.33) and (3.28), we get

$$B = \left\{\left[\tfrac{1}{2} b_I e^{iX} - \left(\frac{igDA^\circ}{2\Omega} b_L\right)\cos K y_1 e^{-i\Omega t_1} + *\right] + i\left(\frac{C_g^\circ}{C_g}\right)^{\frac{1}{2}} b^i\right\} e^{i\theta}, \tag{3.37a}$$

where

$$b_I = \left(\frac{C_g^\circ}{C_g}\right)^{\frac{1}{2}} b^\circ \tag{3.37b}$$

and

$$b_L = \left(\frac{C_g^\circ}{C_g}\right)^{\frac{1}{2}} e^{iX} \int_{-\infty}^{x_1} e^{-iX(\xi)} \frac{W}{C_g} dx_1. \tag{3.37c}$$

The part b_I represents the second-order incident modulation superposed on the first-order amplitude $A(x_1, x_2, \ldots, t_2, t_3)$. The initial amplitude $b^\circ(t_2, t_3, \ldots)$ at $x_1 \sim -\infty$ should be prescribed. On the other hand, b_L is the localized part of B. Being associated with the product DA°, b_L represents the second-order interaction of short swells and long waves. To summarize, the boundary value of the incident swell amplitude is

$$A + \epsilon B \to A^\circ + \epsilon b^\circ \cos Ky_1 \exp i\left[\frac{\Omega}{C_g^\circ}x_1 - \Omega t_1\right] \quad (x_1 \sim -\infty), \tag{3.38}$$

where A° and b° are functions of $t_2, t_3, \ldots$.

3.4. *The long-scale motion* ϕ_{10}

Using (2.13*a, b*) and the fact that

$$\begin{aligned} G_{30} = &-(\phi_{10})_{t_1t_1} - 2(\phi_{00})_{t_1t_2} - \tfrac{1}{2}[(\nabla_1\phi_{00})^2]_{t_1} - [(\phi_{00})_{x_1}(\phi_{00})_{t_1}]_{x_1} \\ &- [(\phi_{00})_{y_1}(\phi_{00})_{t_1}]_{y_1} + \{-[k^2|\phi_{11}|^2 + |(\phi_{11})_z|^2]_{t_1} \\ &+ \frac{1}{g}[\omega^2\phi_{11}(\phi_{11}^*)_z + *]_{t_1} + [2k\omega|\phi_{11}|^2]_{x_1}\}_{z=0}, \end{aligned}$$

we obtain from VSC (2.15)

$$\begin{aligned} (\phi_{10})_{t_1t_1} - g\nabla_1 . (h\nabla_1\phi_{10}) = &-2(\phi_{00})_{t_1t_2} + \frac{g^2}{2\omega}(k|A|^2)_{x_1} - \tfrac{1}{2}[(\nabla_1\phi_{00})^2]_{t_1} \\ &- [(\phi_{00})_{x_1}(\phi_{00})_{t_1}]_{x_1} - [(\phi_{00})_{y_1}(\phi_{00})_{t_1}]_{y_1}. \end{aligned} \tag{3.39}$$

Clearly the right-hand side above contains first harmonics in Ω; they are resonance-forcing (secular) and may be summarized in the form

$$\text{r.h.s. of (3.39)} = R_{10}\cos Ky_1 e^{-i\Omega t_1} + * + \text{NST}, \tag{3.40}$$

where NST stands for non-secular terms. The coefficient R_{10} is simply

$$R_{10} = g\frac{\partial D}{\partial t_2}L_\nu. \tag{3.41}$$

The potential ϕ_{10} may also be expressed as

$$\phi_{10} = \Gamma_{10}(x_1)\cos Ky_1 e^{-i\Omega t_1} + * + \text{NST}, \tag{3.42}$$

where Γ_{10} is the response to R_{10}, satisfying the inhomogeneous equation

$$(h\Gamma_{10}')' + \frac{1}{g}(\Omega^2 - ghK^2)\Gamma_{10} = -\frac{1}{g}R_{10}. \tag{3.43}$$

By applying Green's formula to L_ν which is a homogeneous solution to (3.43) we get a horizontal solvability condition (HSC) for Γ_{10}, i.e.

$$\int_{-\infty}^{\infty} dx_1 R_{10} L_\nu = 0. \tag{3.44}$$

A similar condition was used by Minzoni & Whitham (1977), and is implicit in Guza & Bowen (1976) in their study of edge-wave resonance by subharmonic excitation. Substituting (3.41) in (3.44), we get

$$\frac{\partial D}{\partial t_2} = 0; \tag{3.45}$$

thus $D = D(t_3, t_4, \ldots)$.

To proceed further for the dependence of D on t_3, it is necessary to solve ϕ_{10} explicitly. The right-hand side of (3.39) contains terms proportional to $e^{\pm i\Omega t_1}$ and $e^{\pm 2i\Omega t_1}$, we therefore expect the same harmonics in the response,

$$\phi_{10} = \phi_{10}^h + \phi_{10}^{(0)} + \phi_{10}^{(2)} + \phi_{10}^{(2)*}, \tag{3.46}$$

where ϕ_{10}^h is a homogeneous solution with amplitude E,

$$\phi_{10}^h = -\frac{igE}{2\Omega} L_\nu \cos Ky_1 e^{-i\Omega t_1} + *, \tag{3.47}$$

while $\phi_{10}^{(0)}$, $\phi_{10}^{(2)}$ and $\phi_{10}^{(2)*}$ are particular solutions proportional to $\exp(i0\Omega t)\exp(-2i\Omega t_1)$ and $\exp(2i\Omega t_1)$ respectively. The term $\phi_{10}^{(0)}$ is governed by

$$-g(h\phi_{10x_1}^{(0)})_{x_1} = \frac{g^2}{2\omega}(k|A|^2)_{x_1}. \tag{3.48}$$

The solution is

$$\frac{\partial \phi_{10}^{(0)}}{\partial x_1} = -\frac{gk|A|^2}{2\omega h}. \tag{3.49}$$

This is the usual mass-flux current well known in Stokes wave theory (Whitham 1962). The forcing terms for $\phi_{10}^{(2)}$ contain terms which are independent of y_1 as well as those proportional to $\cos 2Ky_1$. Thus we write

$$\phi_{10}^{(2)} = \left(\frac{-igD}{2\Omega}\right)^2 [\psi_1(x_1) + \psi_2(x_1)\cos 2Ky_1]\, e^{-2i\Omega t_1}. \tag{3.50}$$

It may then be found that

$$-g(h\psi_1')' - 4\Omega^2\psi_1 = \frac{i\Omega}{2}[L_\nu'^2 + K^2L_\nu^2 + \tfrac{1}{2}(L_\nu^2)''] \tag{3.51a}$$

and

$$-g(h\psi_2')' + 4(ghK^2 - \Omega^2)\psi_2 = \tfrac{1}{2}i\Omega[L_\nu'^2 - 3K^2L_\nu^2 + \tfrac{1}{2}(L_\nu^2)''], \tag{3.51b}$$

where primes denote derivatives with respect to x_1. The boundary conditions are

$$\left.\begin{matrix}\psi_1\\ \psi_2\end{matrix}\right\}\text{ bounded everywhere and }\left\{\begin{matrix}\text{outgoing}\\ \downarrow 0\end{matrix}\right\}\quad\text{as}\quad |x_1|\uparrow\infty. \tag{3.52}$$

The vanishing of ψ_2 at $|x_1| \sim \infty$ is dictated by (3.8). For general $h(x_1)$, ψ_1 and ψ_2 may be solved by the numerical method described in appendix A. The solution for ϕ_{10} is now complete for the case of submerged ridge.

To modify the preceding analysis for the closed plane beach we assume that the long wave does not break and replace the integration limits of (3.44) by $(-\infty, 0)$. Since R_{10} involves only the long waves (cf. (3.41)), breaking does not yet have any direct influence on the HSC and (3.45) still holds. However, (3.48) and (3.49) are meaningful only in the shoaling zone and modifications are needed in the surf zone, as will be introduced later. Now ψ_1 and ψ_2 can be solved analytically. In this paper, only the lowest edge wave mode $L_0 = e^{Kx_1}$ will be considered. The solution for ψ_1 is straightforward (Guza & Bowen 1976)

$$\psi_1 = \frac{-2i\pi\Omega K}{g}[(Z_2(\infty) - iZ_1(\infty))\, J_0(4(-Kx_1)^{\frac{1}{2}}) - Z_2(4(-Kx_1)^{\frac{1}{2}}) \\ \times J_0(4(-Kx_1)^{\frac{1}{2}}) + Z_1(4(-Kx_1)^{\frac{1}{2}})\, Y_0(4(-Kx_1)^{\frac{1}{2}})], \tag{3.53a}$$

where

$$Z_1(4(-Kx_1)^{\frac{1}{2}}) = \int_{Kx_1}^{0} e^{2\zeta} J_0(4(-\zeta)^{\frac{1}{2}})\, d\zeta, \tag{3.53b}$$

$$Z_2(4(-Kx_1)^{\frac{1}{2}}) = \int_{Kx_1}^{0} e^{2\zeta} Y_0(4(-\zeta)^{\frac{1}{2}})\, d\zeta. \tag{3.53c}$$

This is an outgoing wave and hence is expected to yield radiation damping, as will be confirmed later. Substituting $L_0 = \exp(Kx_1)$, the right-hand side of (3.51*b*) vanishes; hence, happily,

$$\psi_2 = 0. \tag{3.54}$$

3.5. *Evolution equation for the long trapped wave*

Now we invoke VSC (2.15) for ϕ_{40}. Defining $\hat{G}_{40}$ by

$$G_{40} = -\frac{\partial^2 \phi_{20}}{\partial t_1^2} + \hat{G}_{40},$$

where G_{40} is a lengthy result recorded in appendix C, we get by substituting (3.2) into (2.14) that

$$N_{t_1 t_1} - g\nabla_1 \cdot (h\nabla_1 N) = \frac{g^2 k}{\omega}\frac{\partial |A|^2}{\partial x_2} - g\nabla_1 \left[\frac{h^3}{6}\nabla_1 \nabla_1^2 \phi_{00} - \frac{h^2}{2g}\nabla_1 (\phi_{00})_{t_1 t_1}\right] + \hat{G}_{40}. \tag{3.55}$$

By examining equation (C 1) it is seen that the right-hand side of (3.55) contains various harmonics in $\Omega(0, 1, 2, 3)$. In particular the first harmonic may be collected in the form (3.40) with R_{10} replaced by R_{20}. After straightforward algebra, the function R_{20} is found to be

$$R_{20} = gL_\nu \left(\frac{\partial D}{\partial t_3} + \frac{\partial E}{\partial t_2}\right) + (\tilde{\gamma}_0 + A^{\circ 2}\tilde{\gamma}_1) D + \tilde{\gamma}_2 |D|^2 D + b^\circ A^\circ \tilde{\gamma}_3, \tag{3.56a}$$

where

$$\tilde{\gamma}_0 = \frac{-ig}{4\Omega}\left[\frac{gh^3 K^2}{3}(L_\nu'' - K^2 L_\nu) + \Omega^2 h^2 K^2 L_\nu + \left\{\frac{gh^3}{3}[-L_\nu''' + K^2 L_\nu'] - \Omega^2 h^2 L_\nu'\right\}'\right], \tag{3.56b}$$

$$\begin{aligned}\tilde{\gamma}_1 = \frac{-ig}{4\Omega}\Bigg[&\frac{|A|^2}{A^{\circ 2}}\left\{i\Omega g\left(\frac{q}{ch^2 q} + \frac{1}{thq}\right)\Theta' + (2\omega^2\Omega^2 - g^2K^2)\frac{\Theta}{\omega} + \left(\frac{A^\circ}{|A|}\right)\frac{i\Omega\omega^2 b_L}{sh^2 q}\right.\\ &\left.+\left(2i\Omega\omega k - \frac{i\Omega}{\omega h} gk\right)L_\nu' + \left(\frac{\omega^4\Omega^2}{g^2} - k^2\Omega^2 + \frac{g^2}{2\omega^2}k^2K^2 - \tfrac{1}{2}\omega^2K^2\right)L_\nu\right\}\\ &+\left\{\frac{|A|^2}{A^{\circ 2}}\left(\frac{\Theta'}{\omega}(g^2 + 2\omega^2 gh) + i\Omega g^2 k\frac{\Theta}{\omega^2} + \frac{2}{\omega}g^2 k b_L\left(\frac{A^\circ}{|A|}\right) + 2i\Omega\omega k L_\nu\right.\right.\\ &\left.\left.-\frac{i\Omega}{\omega h}gkL_\nu - \left(\frac{3g^2k^2}{2\omega^2} - \frac{\omega^2}{2}\right)L_\nu'\right)\right\}'\Bigg],\end{aligned} \tag{3.56c}$$

$$\begin{aligned}\tilde{\gamma}_2 = \frac{-ig^3}{8\Omega^3}\{&i\Omega[K^2 L_\nu(3\psi_2 - 2\psi_1) + \psi_1' L_\nu' + \tfrac{1}{2}\psi_2' L_\nu'\\ &+ (2L_\nu'(\psi_1 + \tfrac{1}{2}\psi_2) - L_\nu(\psi_1' + \tfrac{1}{2}\psi_2'))']\\ &+ \frac{3K^2}{8}(L_\nu L_\nu'^2 + 3K^2 L_\nu^3) - \tfrac{3}{8}(3L_\nu'^3 + K^2 L_\nu' L_\nu^2)'\},\end{aligned} \tag{3.56d}$$

$$\tilde{\gamma}_3 = \frac{i\Omega\omega^2}{4sh^2 q}e^{iX}\left(\frac{|A| b_I}{A^\circ b^\circ}\right) + \frac{g^2}{2\omega}\left[k\left(\frac{|A| b_I}{A^\circ b^\circ}\right)e^{iX}\right]'. \tag{3.56e}$$

Invoking HSC on (3.55), we get

$$\left.\begin{aligned} \int_{-\infty}^{\infty} R_{20} L_\nu \, dx_1 &= 0 \quad \text{for a ridge} \\ \text{or} \qquad \int_{-\infty}^{0} R_{20} L_\nu \, dx_1 &= 0 \quad \text{for a beach.} \end{aligned}\right\} \tag{3.57}$$

In either case an evolution equation is obtained at last:

$$\frac{\partial D}{\partial t_3} + \frac{\partial E}{\partial t_2} + (\gamma_0 + A^{\circ 2}\gamma_1) D + \gamma_2 |D|^2 D = -A^\circ b^\circ \gamma_3, \tag{3.58}$$

where

$$\gamma_i = \frac{1}{C_0} \int_{-\infty}^{0,\,\infty} L_\nu \tilde{\gamma}_i \, dx_1 \quad (i = 0, 1, 2, 3) \tag{3.59a}$$

and

$$C_0 = g \int_{-\infty}^{0,\,\infty} L_\nu^2 \, dx_1. \tag{3.59b}$$

The upper limits of (3.59*a*, *b*) are ∞ for a submerged ridge, and 0 for a beach. Physically, γ_0 is derived from the terms in the bracket of (3.55) and represents the correction for linear dispersion of the long waves. Being purely imaginary, γ_0 only affects the phase. γ_1 is associated with the product D and $(A^\circ)^2$, and hence represents the third-order interaction of long and short waves. γ_2 represents the nonlinear interaction between the first and second harmonics of long waves; in particular it contains the effect of the radiated long waves ψ_1. γ_3 is a measure of resonant-forcing and arises from the second-order modulation b° of the incident swell envelope A°.

If the inputs A° and b° do not depend on t_2, the resulting equation resembles that of Minzoni & Whitham (1977). In their evolution equation the term corresponding to the interaction of the edge wave D and the swell A° is of the form $\gamma_1 A^\circ D$, i.e. linear in A°, and is derived at the solvability of the *third*-order problem. In addition, multiple scales exist there for time only but not for space. These factors imply a much less cumbersome algebra in reaching their evolution equation, especially if Airy's shallow-water theory is used from the start (Rockliff 1978). It may also be remarked that the analysis of King & Smith (1978) does not reach high enough order to include the interaction terms γ_1 and γ_2 in (3.58).

4. Trapped waves over a submarine ridge

We first restrict our attention to the ridge without wave breaking. All the integrals involved in (3.59) may be evaluated numerically, yielding the coefficients γ_i, $i = 0$, 1, 2, 3, which are pure constants. Now define $\langle\ \rangle$ to be the time average on the scale t_2 (i.e. storm surge time scale)

$$\langle f \rangle = \lim_{T_2 \to \infty} \frac{1}{T_2} \int_{t_2}^{t_2+T_2} f \, dt_2 \tag{4.1}$$

with

$$(\epsilon^2 \omega)^{-1} \ll T_2 \ll (\epsilon^3 \omega)^{-1}.$$

Thus $\langle f\rangle$ depends at most on $t_3, t_4, \ldots$. Since ϕ_{00} (cf. (3.5)) and ϕ_{10}^h (cf. (3.47)) are of the same form, there is no loss of generality in letting D have all the dependence on t_3 and E depend on t_2 only. Upon taking the average of (3.58) we get

$$\frac{\partial D}{\partial t_3}+(\gamma_0+\langle A^{\circ 2}\rangle\gamma_1)\,D+\gamma_2|D|^2D = -\langle A^\circ b^\circ\rangle\gamma_3. \tag{4.2}$$

Writing

$$A^\circ = \langle A^\circ\rangle+\Delta(t_2), \tag{4.3a}$$

$$b^\circ = \langle b^\circ\rangle+\delta(t_2), \tag{4.3b}$$

it is easily shown that

$$\langle A^{\circ 2}\rangle = \langle A^\circ\rangle^2+\langle\Delta^2\rangle \tag{4.3c}$$

and

$$\langle A^\circ b^\circ\rangle = \langle A^\circ\rangle\langle b^\circ\rangle+\langle\Delta\delta\rangle. \tag{4.3d}$$

Subtraction of (4.2) from (3.58) gives

$$\frac{\partial E}{\partial t_2} = -2\langle A^\circ\rangle\,\Delta\gamma_1 D-[\Delta\langle b^\circ\rangle+\delta\langle A^\circ\rangle]\,\gamma_3. \tag{4.4}$$

Thus by prescribing the envelope modulations and D at $t_3 = 0$, (4.2) may be integrated numerically for $D(t_3)$. Afterwards (4.4) is integrated by quadrature with respect to t_2 for $E(t_2)$ with zero initial value.

We observe first that, if the modulation depends on t_3 but not t_2, $\Delta = \delta = 0$, then E vanishes. Otherwise E is always bounded because $\langle\Delta\rangle = \langle\delta\rangle = 0$. Hence we shall focus attention to D only from here on. On the other hand, even if $\langle A^\circ\rangle$ or $\langle b^\circ\rangle$ vanishes there is still resonant forcing as long as the t_2-fluctuations of the first-order envelope and of the second-order modulation have non-zero correlation over t_3 scale $\langle\Delta\delta\rangle \neq 0$. Thus storm surges with 1–5 min modulation can also excite trapped waves.

Multiplying (4.2) by D^* and adding the resulting equation to its own complex conjugate, we get

$$\frac{\partial|D_0|^2}{\partial t_3}+2\langle A^{\circ 2}\rangle\,\mathrm{Re}\,\gamma_1|D|^2+2\,\mathrm{Re}\,\gamma_2|D|^4 = -\langle A^\circ b^\circ\rangle 2\,\mathrm{Re}\,(\gamma_3 D_0); \tag{4.5}$$

use has been made of the fact that γ_0 is imaginary. The above equation may be interpreted as the energy equation for the long waves. The right-hand side is the rate of working by resonant forcing. On the left-hand side, the second term gives the effect of damping if $\mathrm{Re}\,\gamma_1 > 0$ and instability if $\mathrm{Re}\,\gamma_1 < 0$. Thus the damping or unstable growth rate rises with the magnitude and duration of $\langle A^\circ\rangle^2$ or its t_2-fluctuation $\langle\Delta^2\rangle$. The third term from nonlinearity also gives rise to damping if $\mathrm{Re}\,\gamma_2 > 0$ which is to be verified. From (3.51a) it is seen that ψ_2 is purely imaginary and does not contribute to $\mathrm{Re}\,\gamma_2$ (see (3.56d)); hence this damping is entirely due to the radiation of second harmonic long waves corresponding to ψ_1. If the initial value of $|D|^2$ is small, the linear (second) term in (4.5) dominates so that energy transfer between long trapped waves and short swells overshadows radiation damping.

If the incident swells and their modulations, and hence $\langle A^\circ b^\circ\rangle$, approach a steady state as $t_3\to\infty$, $|D|$ can also approach a finite limit which may be obtained as follows. Letting

$$\left.\begin{aligned}&\frac{\partial}{\partial t_3} = 0, \quad V_1 = \gamma_0+\langle A^{\circ 2}\rangle\gamma_1,\\ &V_2 = \gamma_2, \quad V_3 = -\langle A^\circ b^\circ\rangle\gamma_3 = |V_3|\,e^{i\theta_3}\end{aligned}\right\} \tag{4.6}$$

in (4.2) we get

$$V_1 D + V_2|D|^2 D = V_3. \tag{4.7}$$

The square of the absolute value of (4.7) is

$$|V_1|^2\,|D|^2 + (\operatorname{Re} V_1 \operatorname{Re} V_2 + \operatorname{Im} V_1 \operatorname{Im} V_2)\,|D|^4 + |V_2|^2\,|D|^6 = |V_3|^2. \tag{4.8}$$

$|D|^2$ is then found as the real positive root of the above cubic equation. Afterwards the phase of D defined by $D = |D| \exp(i(\theta_D + \theta_3))$ may be found from

$$\theta_D = -\arg(V_1 + V_2|D|^2). \tag{4.9}$$

The numerical examples are presented in dimensionless variables normalized by the length scale $1/k^\circ$ and the time scale $1/\omega$, or specifically

$$\left.\begin{aligned}(k, K) &= k^\circ(\bar{k}, \bar{K}), \quad \Omega = \omega\Omega, \quad (x_1, h) = (\bar{x}_1, \bar{h})/k^\circ,\\ (t_2, t_3) &= (\bar{t}_2, \bar{t}_3)/\omega, \quad (A^\circ, b^\circ, D) = (\bar{A}^\circ, \bar{b}^\circ, \bar{D})/k^\circ.\end{aligned}\right\} \tag{4.10}$$

A symmetric ridge of the following depth profile is assumed

$$\bar{h} = \bar{h}^\circ[1-(1-r)\exp(-\beta\bar{x}_1^2)] \tag{4.11}$$

where $\bar{h}^\circ$ is the dimensionless depth at infinity, and $r\bar{h}^\circ$ the minimum depth at $\bar{x}_1 = 0$. The coefficient β is set to be

$$\beta = e/2[(1-r)\bar{h}^\circ]^2 \tag{4.12}$$

so that the maximum value of bottom slope is always unity (ϵ in physical variables). A wave packet is incident from $\bar{x}_1 \sim -\infty$.

First we choose the modal number ν; then, for a given set of $\bar{h}^\circ$, r, and $\bar{\Omega}$, the modal shape of the eigenfunction $L_\nu(\bar{x}_1)$ and the eigenvalue K are found by the numerical method of appendix A. The coefficients γ_0, γ_1, γ_2 and γ_3 are then calculated by Simpson's rule. Sample results for the lowest even mode $\nu = 0$ are listed in table 1.

From table 1, it can be seen that $\operatorname{Re}\gamma_1 > 0$ always; therefore, by numerical evidence the incident swells *extract* energy from the trapped long waves and the resonant modulation b° is necessary for exciting the latter. Also for the same input parameters the cubic equation (4.8) has only one real solution, hence only one steady state.

In figure 1 we show the approach to a steady state by letting $\bar{A}^\circ = \bar{A}_0$ be a constant and $\bar{b}^\circ$ be a step function: $\bar{b}^\circ = \bar{A}_0 H(t_3)$. The initial value of $\bar{D}$ is zero. The results for several values of $\bar{A}_0$ show a rapid rise followed by an oscillatory approach to the steady state. The final limits are not monotonic in $\bar{A}_0$ and have been checked with the solution of (4.8).

Figure 2 shows the response to transient inputs of duration 6. In the solid curve (1) both $\bar{A}^\circ$ and $\bar{b}^\circ$ are limited in duration T according to

$$\bar{A}^\circ = \bar{b}^\circ = \exp[-(18/T^2)(\bar{t}_3 - \tfrac{1}{2}T)^2].$$

For the solid curve (2) the resonant modulation $\bar{b}^\circ$ is short-lived as given above but the swells are persistent: $\bar{A}^\circ = 1$. In the former case only the weak radiation damping is in effect after the passing of the forcing. In the latter case the trapped waves experiences an additional energy drain to the long-lived swells. In figure 3 similar results are presented by solid curves with the duration reduced to 2. The peak responses are now much reduced and the attenuation is also weak because of the small amplitude. These results suggest that friction at the sea bottom should also be

$\bar{h}^{\circ}$	r	$\bar{\Omega} = 1{\cdot}0$					$\Omega = 2{\cdot}0$				
		$\bar{K}$	γ_0	γ_1	γ_2	γ_3	$\bar{K}$	γ_0	γ_1	γ_2	γ_3
1	0·5	0·860	$-0{\cdot}121i$	$0{\cdot}09+0{\cdot}10i$	$0{\cdot}009-0{\cdot}30i$	$-0{\cdot}01-0{\cdot}002i$	1·83	$-0{\cdot}91i$	$0{\cdot}36+0{\cdot}195i$	$0{\cdot}04+0{\cdot}40i$	$0{\cdot}065+0{\cdot}28i$
1	0·3	0·917	$-0{\cdot}116i$	$0{\cdot}44+0{\cdot}21i$	$0{\cdot}022-0{\cdot}45i$	$-0{\cdot}26-0{\cdot}029i$	2·00	$-0{\cdot}73i$	$6{\cdot}94-2{\cdot}59i$	$0{\cdot}10-0{\cdot}03i$	$-1{\cdot}65-0{\cdot}88i$
1	0·1	1·13	$-0{\cdot}113i$	$13{\cdot}2-3{\cdot}1i$	$0{\cdot}161-0{\cdot}20i$	$-2{\cdot}40-0{\cdot}532i$	3·31	$-0{\cdot}37i$	$112{\cdot}3-19{\cdot}3i$	$1{\cdot}20-3{\cdot}01i$	$-11{\cdot}15-4{\cdot}08i$
2	0·5	0·713	$-0{\cdot}354i$	$0{\cdot}01+0{\cdot}08i$	$0{\cdot}007-0{\cdot}09i$	$0{\cdot}005-0{\cdot}015i$	1·43	$-2{\cdot}0i$	$0{\cdot}033+0{\cdot}08i$	$0{\cdot}01-0{\cdot}20i$	$0{\cdot}11-0{\cdot}024i$
2	0·3	0·786	$-0{\cdot}255i$	$0{\cdot}197+0{\cdot}078i$	$0{\cdot}013-0{\cdot}05i$	$0{\cdot}111-0{\cdot}106i$	1·8	$-1{\cdot}25i$	$1{\cdot}2+1{\cdot}77i$	$0{\cdot}03-0{\cdot}037i$	$-0{\cdot}123-0{\cdot}6i$
2	0·1	1·10	$-0{\cdot}177i$	$2{\cdot}43+0{\cdot}44i$	$0{\cdot}046-0{\cdot}02i$	$0{\cdot}755+0{\cdot}030i$	3·25	$-0{\cdot}41i$	$23{\cdot}9+0{\cdot}61i$	$0{\cdot}42+0{\cdot}05i$	$-3{\cdot}52-0{\cdot}09i$
3	0·5	0·60	$-0{\cdot}931i$	$0{\cdot}008-0{\cdot}05i$	$0{\cdot}001+0{\cdot}008i$	$0{\cdot}009+0{\cdot}007i$	1·32	$-2{\cdot}85i$	$0{\cdot}001+0{\cdot}016i$	$0{\cdot}001-0{\cdot}10i$	$0{\cdot}016+0{\cdot}011i$
3	0·3	0·68	$-0{\cdot}566i$	$0{\cdot}079+0{\cdot}07i$	$0{\cdot}009-0{\cdot}027i$	$-0{\cdot}001-0{\cdot}076i$	1·61	$-1{\cdot}67i$	$0{\cdot}091+0{\cdot}52i$	$0{\cdot}02-0{\cdot}07i$	$0{\cdot}071-0{\cdot}13i$
3	0·1	1·08	$-0{\cdot}231i$	$0{\cdot}20+0{\cdot}13i$	$0{\cdot}025-0{\cdot}134i$	$-0{\cdot}019-0{\cdot}002i$	2·80	$-0{\cdot}52i$	$9{\cdot}82+3{\cdot}58i$	$0{\cdot}13-0{\cdot}08i$	$1{\cdot}93-0{\cdot}93i$

TABLE 1. Coefficients of the trapped wave equation for several ridges. Only the lowest mode $\nu = 0$ is considered.

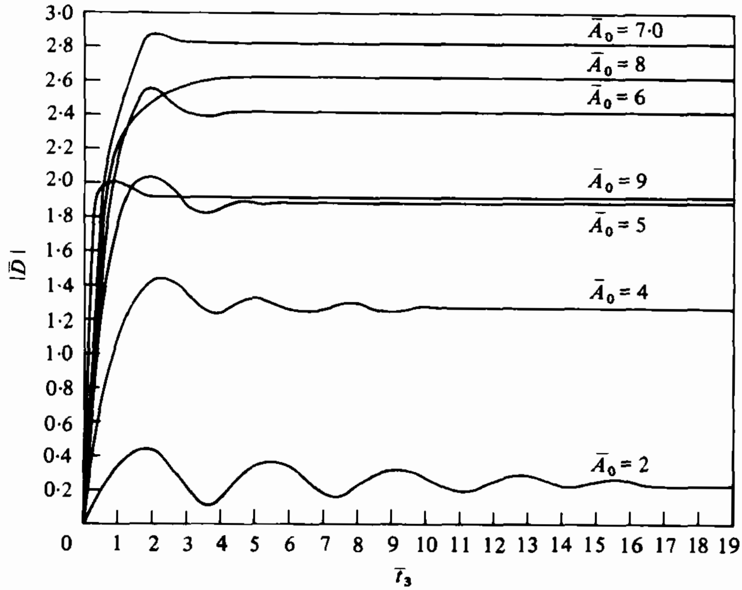

FIGURE 1. Transient evolution of trapped waves due to step-function forcing $\bar{A}^\circ = \bar{A}_0$ and $\bar{b}^\circ = \bar{A}_0 H(\bar{t}_3)$. $\bar{\Omega} = 2$, $\bar{h}^\circ = 2$ and $r = 0{\cdot}5$.

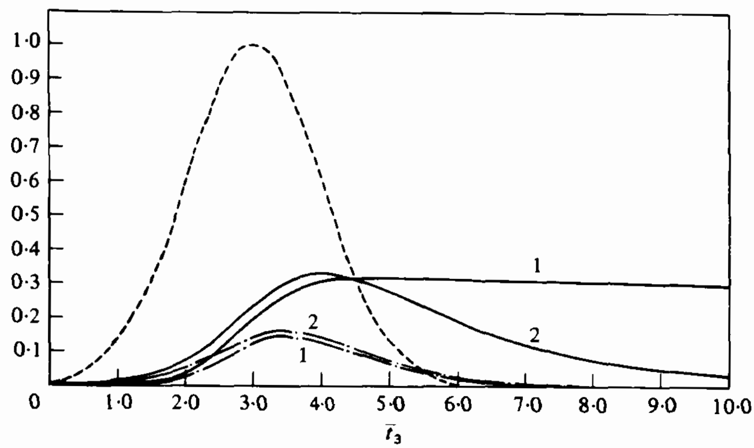

FIGURE 2. Transient response of the trapped wave amplitude $|D|$, over a submerged ridge, due to a short-lived resonant modulation $\bar{b}^\circ$ to the incident swells amplitude $\bar{A}^\circ$. ----, $\bar{b}^\circ = \exp[-0{\cdot}5(\bar{t}_3 - 3)^2]$; ——, $|D|$ without bottom friction; —·—, $|D|$ with bottom friction. Curve 1 refers to the case $\bar{A}^\circ = \bar{b}^\circ$, i.e. the wave packet is also short-lived. Curve 2 to $\bar{A}^\circ = 1{\cdot}0$ i.e. a long-lived wave packet. In all cases $\bar{\Omega} = 1$, $\bar{h}^\circ = 1$ and $r = 0{\cdot}3$.

included. By assuming a bottom stress linear in the local velocity we can add an empirical constant to the first bracket in the evolution equation (4.2). The rationale of this artifice is given in appendix D. Using some reasonable numbers the bottom friction damping time is estimated to be $O(1 \text{ day})$. With this, the computations corresponding to the earlier inputs were then repeated. The results shown by chain lines in figures 2 and 3 indicate a markedly faster attenuation of D, accompanied by a lowering of the peak as well as the shortening of the lag time between the peaks of input and output.

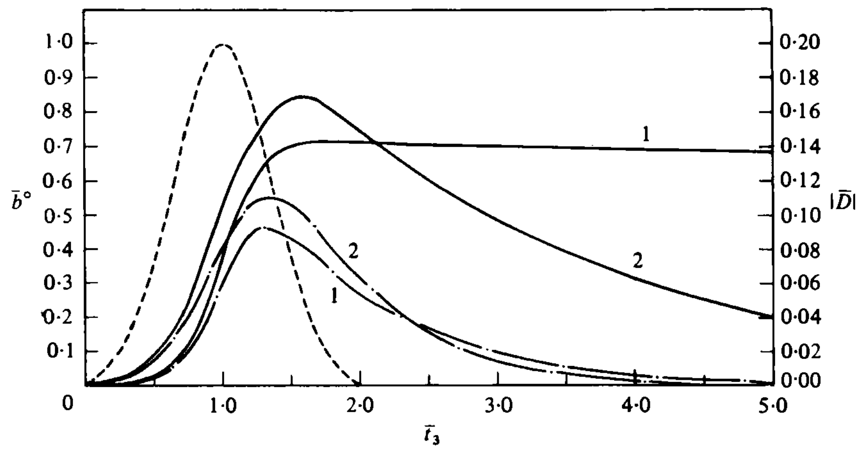

FIGURE 3. Same description as for figure 2, except that the resonant modulation $\bar{b}^\circ$ has shorter duration, $\bar{b}^\circ = \exp[-4\cdot0(\bar{t}_3-1)^2]$. Note that the scale for $|D|$ is to the right.

5. Surf beats on a plane beach

We shall assume that the long wave does not break and the short swells break completely upon crossing the breaker line. The effect of breaking on both will be assessed by invoking some empirical relations for the short swells. Let us denote the breaker line by $x_1 = x_{1b} < 0$ determined by the empirical formula that

$$|2\epsilon A|/h = \gamma_b = 0\cdot7 \sim 1\cdot2. \tag{5.1}$$

Strictly speaking the total amplitude and the instantaneous sea depth should be used in (5.1) and a nonlinear theory is needed. In view of the empirical uncertainties the simpler linear theory is used here. By combining (3.16) with (5.1) and assuming that $kh \ll 1$ at x_{1b} we obtain

$$-\bar{x}_{1b} = \left(\frac{\sqrt{2}}{\gamma_b}\epsilon k^\circ A^\circ\right)^{\frac{4}{5}}, \tag{5.2}$$

where $(\ )^\circ$ corresponds to infinite depth relative to the swell length. Since the magnitude $|A|$ of the local amplitude does not depend on t_1 the breaker line does not oscillate with the trapped wave. The t_1-oscillation of the trapped waves induces oscillations in the phase of A, implying that swells do not begin to break at the same instants within a period $2\pi/\Omega$ though approximately at the same distance from shore. Note that $\bar{x}_{1b}$ is small because of ϵ. The use of linear theory is of course not legitimate but Komar & Gaughan (1972) found that (5.2) agrees with observations if $\gamma_b \simeq 1\cdot4$. Here we shall ignore this discrepancy with (5.1) because of the smallness of $\bar{x}_{1b}$. Within the surf zone it is usually assumed that the local breaking-wave amplitude and the local mean depth are also related by (5.1), where h should be replaced by $h+\epsilon\eta_{10}$. Ignoring $\epsilon\eta_{10}$ relative to h the amplitude A then diminishes linearly with depth or $(-\bar{x}_1)$, i.e.

$$\frac{|A|}{A^\circ} = -s\hat{x}_1 \quad \text{or} \quad \frac{A}{A^\circ} = (-sx_1)\,e^{i\theta} \quad (\bar{x}_1 < \bar{x}_{1b}). \tag{5.3}$$

By matching the amplitudes at the breaker line, the coefficient s is found:

$$s = \frac{1}{\sqrt{2}}(-\bar{x}_{1b})^{-\frac{5}{4}}. \tag{5.4}$$

Additional hypothesis is needed for the amplitude of the resonant modulation b_I in the breaking zone. Note from (3.37 *b*) that

$$\frac{b_I}{|A|} = \frac{b^\circ}{A^\circ} \tag{5.5}$$

in the shoaling zone; we shall assume that the same relation holds in the surf zone, implying that

$$b_I = (-s\bar{x}_1)\, b^\circ, \quad \bar{x}_1 < \bar{x}_{1b}. \tag{5.6}$$

Moreover, we assume that all results in § 3 still hold formally. However, referring to (3.59) the integrals in γ_1 and γ_3 involve swells and must be broken into two parts. In the surf zone (5.3)–(5.6) are imposed. Outside the surf zone the potential theory is still used. The resulting γ_1 and γ_3 are functions of $t_2, t_3, \ldots$ through their dependence on the breaker line location $\bar{x}_{1b}$ and hence on A°. On the other hand the coefficients γ_0 and γ_2 involve only long waves and can be evaluated straightforwardly to yield constant values. Thus through γ_1 and γ_3 the long wave is indirectly affected by breaking.

Again assuming that the amplitude of the second-order edge waves E depends on t_2 only, (3.58) may be averaged over t_2, yielding

$$\frac{\partial D}{\partial \bar{t}_3} + [\gamma_0 + \langle A^{\circ 2}\gamma_1\rangle]\, D + \gamma_2 |D|^2 D = -\langle A^\circ b^\circ \gamma_3\rangle. \tag{5.7}$$

Note that γ_1 and γ_3 depend on t_2 and t_3 ... only through A° but not through the resonant modulation b°. Thus, in the special case where A° does not depend on t_2, one may remove the averaging brackets and replace b° by its average $\langle b^\circ\rangle$. For simplicity only this special case is dealt with in the ensuing computations. All the integrals in γ_i, $i = 0, 1, 2, 3$, are evaluated numerically for the lowest edge wave mode $\nu = 0$. In figures 4 (*a*, *b*) we show γ_1 as a function of $\epsilon k^\circ A^\circ$ and $\bar{\Omega}$. In all cases $\mathrm{Re}\,\gamma_1 < 0$, implying *instability*, in contrast with the submarine ridge. Furthermore from the figure an empirical formula may be obtained

$$\mathrm{Re}\,\bar{\gamma}_1 = F(\bar{\Omega})/(\epsilon k^\circ A^\circ)^{\frac{4}{3}} \tag{5.8}$$

with $F(\bar{\Omega})$ listed in table 2. It can also be seen that for $\bar{\Omega} > 3$ the surf zone prevails over the shoaling zone in contributing to the growth rate. Note also that as $k^\circ A^\circ \downarrow 0$ the term $(\bar{A}^\circ)^2\gamma_1 \downarrow 0$ also. Because of instability it is in principle not necessary to have resonant modulation for the excitation of trapped waves if the initial value of $\bar{D}$ is not zero. A sample result without resonant forcing is shown by the solid curve (2) in figure 6 for a Gaussian swell packet of duration 2. However, the growth rate is weak so that radiation damping prevents the small initial disturbance from getting large. To achieve a significant amplitude, the edge wave must be forced by resonant modulation. Calculated values of γ_3 are shown in figures 5 (*a*, *b*). In figure 6, the solid curve (1) shows the response to a transient wave packet subject to a transient resonant modulation. Similar computations for a longer duration (~ 6) without bottom friction are shown in figure 7. Using the same empirical estimate as in § 5 the effect of bottom friction is seen to overwhelm the instability growth from the start as shown by the chain lines in figure 6. Clearly instability alone cannot overcome friction. With

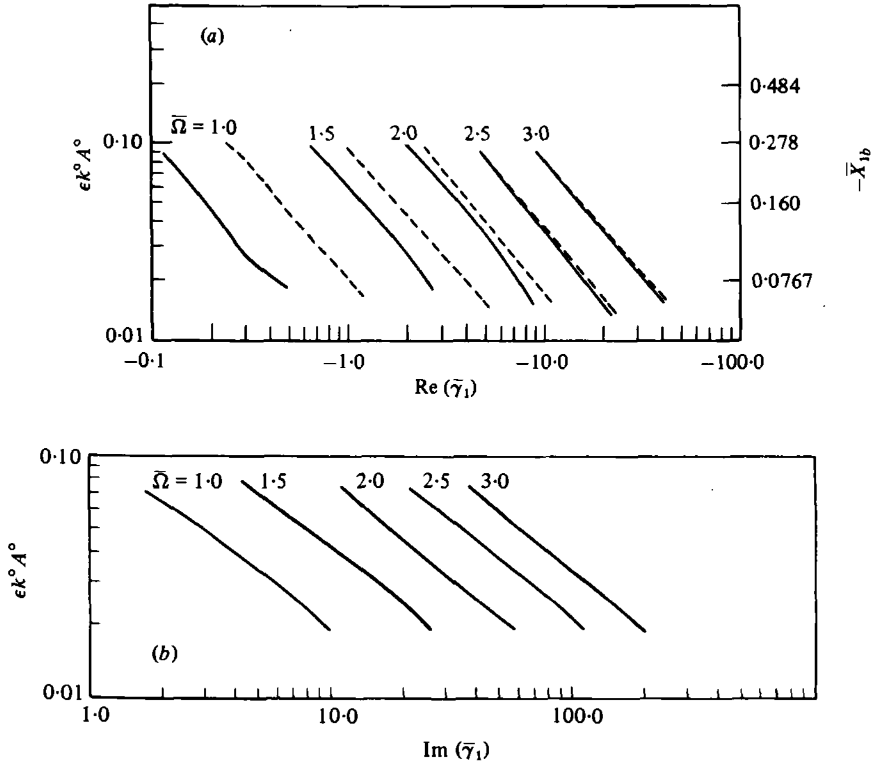

FIGURE 4. (*a*) Re $(\bar{\gamma}_1)$ = Re $(\gamma_1)/\omega k^{\circ 2}$ *vs*. $\epsilon k^\circ A^\circ$ for different values of $\bar{\Omega}$. ——, total Re $(\bar{\gamma})$; ----, contribution from surf zone. $\gamma_b = 0{\cdot}7$ (cf. (5.1)).
(*b*) Im $(\bar{\gamma}_1)$ = Im $(\gamma_1)/\omega k^{\circ 2}$ *vs*. $\epsilon k^\circ A^\circ$ for different values of $\bar{\Omega}$.

Ω	1	1·5	2·0	2·5	3·0
$F(\Omega)$	−0·016	−0·105	−0·328	−0·683	−1·365

TABLE 2. Empirical coefficient $F(\Omega)$ in equation (5.8).

resonant forcing, the bottom friction reduces the height of and shifts the response peak towards the peak of the wave packet. These results are in order-of-magnitude accord with the observations of Munk and Tucker.

Finally, if the first-order envelope A° approaches a steady state, then a steady response may be established by the equilibrium between instability and nonlinear radiation damping, after the forcing expires. The equilibrium limit is

$$|D|^2 = \frac{(-\operatorname{Re}\gamma_1)A^{\circ 2}}{\operatorname{Re}\gamma_2}, \quad t_3 \to \infty, \tag{5.9}$$

from (4.5). For $\nu = 0$, $\operatorname{Re}\gamma_2$ may be evaluated analytically by using (3.53),

$$\operatorname{Re}\gamma_2 = 4\pi Z_1(\infty)\,\Omega^5/g^2 = 0{\cdot}0575\Omega^5/g^2. \tag{5.10}$$

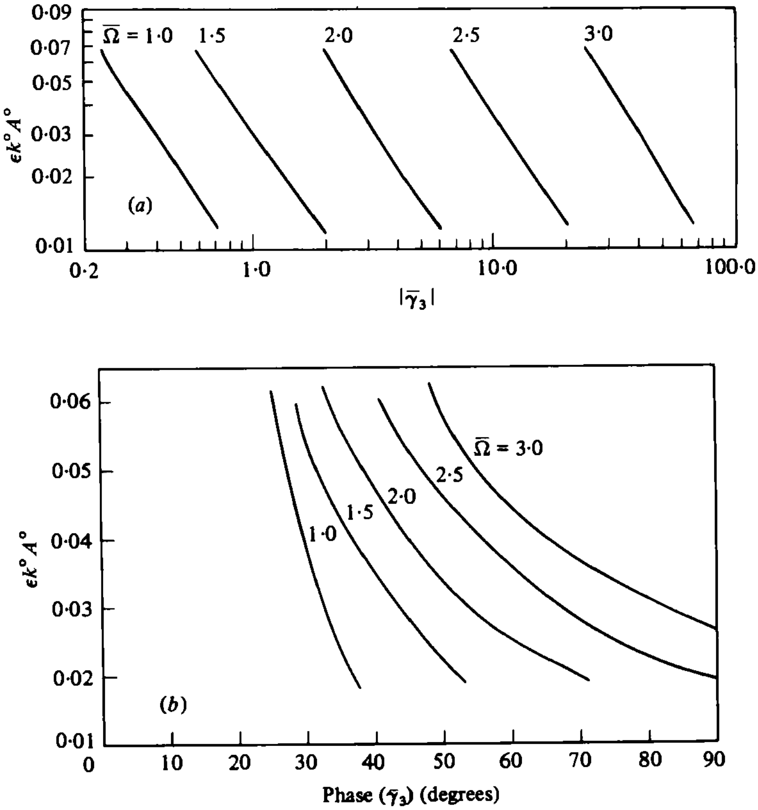

FIGURE 5. (a) Absolute value of $\bar{\gamma}_3 = |\gamma_3|/\omega k^\circ$ *vs.* $\epsilon k^\circ A^\circ$ for different values of $\bar{\Omega}$. (b) Change of phase of γ_3 with $\epsilon k^\circ A^\circ$ and $\bar{\Omega}$.

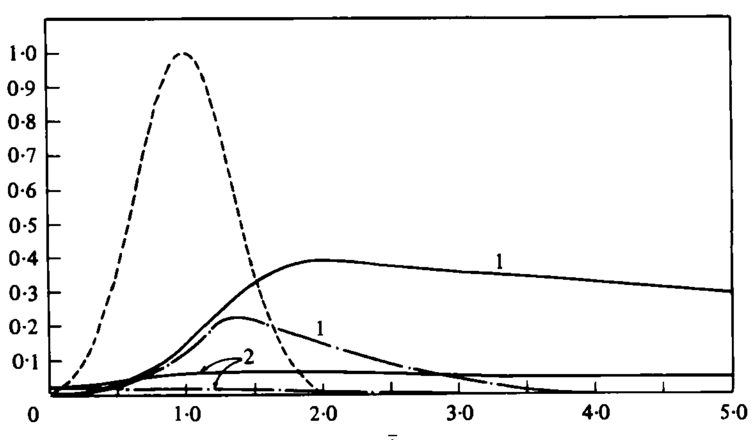

FIGURE 6. Transient response of the edge wave amplitude $|D|$, over a plane beach, due to a short-lived incident wave packet. ----, $\bar{A}^\circ = \exp[-4{\cdot}0(\bar{t}_3-1)^2]$, ——, $|D|$ without friction; —·—, $|D|$ with friction. Curve 1 refers to the case of a resonantly modulated wave with $\bar{b}^\circ = \bar{A}^\circ$; curve 2 to $\bar{b}^\circ = 0$ (no resonant modulation). $\bar{\Omega} = 1$.

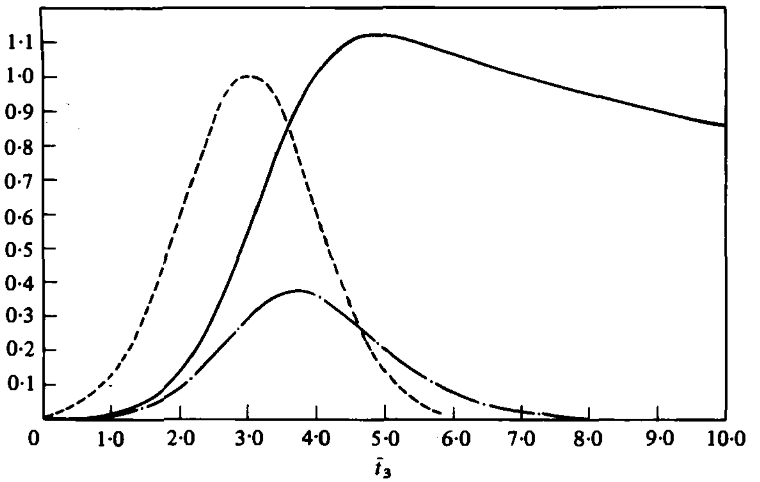

FIGURE 7. Same description as for figure 6 except that the incident wave has a longer life: $\bar{A}^\circ = \exp[-0{\cdot}5(\bar{t}_3 - 3)^2]$. Only the resonant modulation case is shown here. The result with $\bar{b}^\circ = 0$ resembles that in figure 6.

Combining (5.8) and (5.10) we have

$$-\frac{\operatorname{Re}\gamma_1}{\operatorname{Re}\gamma_2} = -\frac{\operatorname{Re}\tilde{\gamma}_1}{\operatorname{Re}\bar{\gamma}_2} = \frac{-F(\Omega)}{(\epsilon k^\circ A^\circ)^{\frac{4}{5}}\,\Omega^5}. \tag{5.11}$$

Since $k^\circ A^\circ = O(1)$ by normalization, we take $\epsilon = 0{\cdot}05$ and use table 2 to get $|D|/A^\circ \approx 1{\cdot}4$–$0{\cdot}24$ for $\Omega \approx 1$–3. In reality the variability in storms limits the practical relevance of the above result.

6. Conclusions

Long-standing waves trapped on a straight beach or over a submarine ridge can be resonated by normally incident swells with slow modulations in time and in the long-shore direction. In both topographies, nonlinear radiation damping by the emission of second harmonics of long waves limits the resonance amplitude. Nonlinear interaction transfers energy from the long wave to the swells in the case of a ridge. This energy transfer lasts only as long as the life span of the wave packet. Bottom friction adds substantial damping, which reduces the resonant peak and expedites attenuation. On a closed plane beach, the breaking swells feed energy to the edge waves, at however a very weak rate which is easily overcome by bottom friction. Thus forcing by resonant modulation in the incident swells is still the most significant mechanism. Controlled experiments are needed.

It is clear that the present analysis can be extended to progressive trapped waves. Furthermore, since a jet-like steady current can be a wave-guide for waves of certain lengths, it is in principle possible to modify the present analysis for the resonance of long waves trapped in the current by normally incident short swells.

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Appendix A. A hybrid element method for long waves on a submerged ridge

Equations (3.6), (3.51*a*) and (3.51*b*), for L_ν, ψ_1 and ψ_2 respectively, have the general non-dimensional form

$$(\bar{h}\Lambda')' + \lambda\Lambda = f(\bar{x}_1), \quad -\infty < \bar{x}_1 < \infty. \tag{A 1}$$

The boundary conditions at $\bar{x}_1 \sim \pm\infty$ are given by (3.52), where λ is a function of $\bar{x}$. To illustrate the numerical procedure for solving (A 1), we consider the inhomogeneous problem for the case of even $\bar{h}$, λ and f. Now only one half of the fluid domain $\bar{x}_1 > 0$ needs to be considered. The function λ is known beforehand. First introduce the vertical line $\bar{x}_1 = l$ which divides the domain into two regions; (1) the region $0 \leqslant \bar{x}_1 \leqslant l$ where the depth $\bar{h}(\bar{x}_1)$ is variable, and (2) $\bar{x}_1 > l$; $\bar{h} = \bar{h}^\circ$ is constant.

In the region of constant depth (outer region), the solution to (A 1) can easily be represented analytically. However, this formal solution can only satisfy the boundary condition (3.52*a* or *b*) at $\bar{x}_1 \sim \infty$; and hence has one yet undetermined coefficient (say, a).

In the region of variable depth, a one-dimensional finite-element discretization is constructed for equation (A 1). Furthermore, two matching conditions between the finite element and the analytical representations are required; they are

$$\left.\begin{aligned} \Lambda^+ &= \Lambda^- \\ \frac{\partial \Lambda^+}{\partial \bar{x}_1} &= \frac{\partial \Lambda^-}{\partial \bar{x}_1} \end{aligned}\right\} \quad \text{at} \quad \bar{x}_1 = l, \tag{A 2}$$

where $\Lambda^+ \equiv$ outer (analytical) representation and Λ^- finite-element representation. In addition the boundary condition at the axis of symmetry for the ridge is

$$\frac{\partial \Lambda^-}{\partial x_1} = 0, \quad x_1 = 0. \tag{A 3}$$

Similar to Chen & Mei (1974), it is straightforward to show that the problem (A 1) – (A 3) is equivalent to the stationarity of the functional

$$J = \int_0^l \left[\frac{\bar{h}}{2}\left(\frac{\partial \Lambda^-}{\partial \bar{x}_1}\right)^2 - \frac{\lambda}{2}\Lambda^{-2} + f\Lambda^-\right] d\bar{x}_1 + \left[\bar{h}\,(\tfrac{1}{2}\Lambda^+ - \Lambda^-)\frac{\partial \Lambda^+}{\partial \bar{x}_1}\right]_{x_1 = l}. \tag{A 4}$$

Note that the integral is over the finite length l. Using the finite-element discretization, we can approximate the expression (A 4) in terms of nodal values of Λ and the analytical solution coefficient (a), yielding the bilinear expression

$$J = \tfrac{1}{2}\mathbf{x}^T[A]\,\mathbf{x} - \mathbf{x}^T\mathbf{F}, \tag{A 5}$$

where $\mathbf{x}$ is the vector of unknowns, $[A]$ a symmetric matrix, and $\mathbf{F}$ is the vector containing the values of (f) at the nodal points. The stationarity of J implies

$$[A]\mathbf{x} = \mathbf{F}. \tag{A 6}$$

This set of algebraic linear equations can be solved by, for example, Gaussian elimination, to get the unknown vector $\mathbf{x}$.

For the homogeneous problem of L_ν (cf. (3.6)), the quantity $\lambda = (\tanh k^\circ h^\circ)\bar{\Omega}^2 - \bar{h}\bar{K}^2$ contains the eigenvalue $\bar{K}$ for a given $\bar{\Omega}$. (In dimensional form $\lambda = (\Omega^2 - ghK^2)/g$.) The boundary condition at the top of the ridge is

$$\left.\begin{aligned} L_\nu' &= 0 \quad \text{for } \nu \text{ even,} \\ \text{or} \qquad L_\nu &= 0 \quad \text{for } \nu \text{ odd,} \end{aligned}\right\} \quad x_1 = 0. \tag{A 7}$$

At $x_1 \sim \pm\infty$, $L_\nu \downarrow 0$, the stationary function is given by (A 4) with $f = 0$. Upon discretization, equation (A 6) holds with $\mathbf{F} = 0$ and the elements of $[A]$ containing the unknown $\bar{K}$ for a fixed Ω. Hence, the resulting set of homogeneous algebraic equations are solved for a given value of Ω by trial and error for the eigenvalues $\bar{K}$. Then the corresponding modal shapes are obtained from (A 6) after suitable normalization (e.g. $L_\nu(\bar{x}_1 = 0) = 1$ for ν = even).

Appendix B. Detailed expression of S

$$\begin{aligned} S = -i\omega \cosh qA \Bigg\{ &-\frac{\alpha_1\alpha_2}{\sinh 2q} - \frac{1}{4}\left(1 + \frac{2q}{\sinh 2q}\right)\beta_1 \\ &- \tfrac{1}{4}\beta_2\left(\coth 2q - \frac{1}{\sinh 2q}\right) - \tfrac{1}{8}\beta_3\left(2q - \coth 2q + \frac{1+2q^2}{\sinh 2q}\right) \\ &- \tfrac{1}{8}\beta_4(2q\coth 2q - 1) + \tfrac{1}{8}\beta_5\left(2q^2 - 2q\coth 2q + 1 + \frac{4q^3}{\sinh 2q}\right) \\ &- \tfrac{1}{8}\beta_6\left[(1+2q^2)\coth 2q - 2q - \frac{1}{\sinh 2q}\right] \\ &- \tfrac{1}{16}\beta_7[(4q^3 + 6q)\coth 2q - (6q^2+3)]\Bigg\} + \frac{i\omega\cosh q}{2k^2}\frac{\partial^2 A}{\partial y_1^2}\left(\frac{1}{2} + \frac{q}{\sinh 2q}\right), \end{aligned} \tag{B 1}$$

where

$$\beta_1 = 3\alpha_1^2 + \alpha_6, \quad \beta_2 = 4\alpha_1\alpha_2 + 4\alpha_1\alpha_3 + \alpha_4, \quad \beta_3 = 14\alpha_1\alpha_3 + 4\alpha_1\alpha_2 + 2\alpha_4,$$

$$\beta_4 = 6\alpha_2\alpha_3 + 2\alpha_1^2 + 2\alpha_5 + 2\alpha_6, \quad \beta_5 = 2\alpha_5 + 6\alpha_2\alpha_3 + 6\alpha_3^2,$$

$$\beta_6 = 6\alpha_1\alpha_3, \quad \beta_7 = 4\alpha_3^2,$$

with

$$\left.\begin{aligned} \alpha_1 = h', \quad \alpha_2 = \frac{(A/\cosh q)'}{k(A/\cosh q)}, \quad \alpha_3 = \frac{k'}{2k^2}, \\ \alpha_4 = \frac{h''}{k}, \quad \alpha_5 = \frac{k''}{2k^3}, \quad \alpha_6 = \frac{(A/\cosh q)''}{k^2(A/\cosh q)}. \end{aligned}\right\}$$

The last term in S is new and does not appear in Chu & Mei (1970).

Appendix C. The secular forcing for the function ϕ_{20}

Lengthy but straightforward algebra leads to the following result:

$$
\begin{aligned}
\hat{G}_{40} = {} & -2\frac{\partial^2\phi_{00}}{\partial t_1\partial t_3} - 2\frac{\partial^2\phi_{10}}{\partial t_1\partial t_2} - \frac{\partial}{\partial t_2}\left[k^2|\phi_{11}|^2 + \left|\frac{\partial\phi_{11}}{\partial z}\right|^2\right] \underline{- \frac{\partial}{\partial t_1}\left[\left(k^2\phi_{21}\phi_{11}^* + \frac{\partial\phi_{21}}{\partial z}\frac{\partial\phi_{11}^*}{\partial z}\right)+*\right]} \\
& -\frac{\partial}{\partial t_1}\left[ik\phi_{11}\frac{\partial\phi_{11}^*}{\partial x_1}+*\right] - \frac{\partial}{\partial t_1}\left[\frac{\partial\phi_{00}}{\partial x_1}\frac{\partial\phi_{10}}{\partial x_1}+\frac{\partial\phi_{00}}{\partial y_1}\frac{\partial\phi_{10}}{\partial y_1}\right] - \frac{1}{2}\frac{\partial}{\partial t_2}\left[\left(\frac{\partial\phi_{00}}{\partial x_1}\right)^2+\left(\frac{\partial\phi_{00}}{\partial y_1}\right)^2\right] \\
& +\frac{1}{g}\frac{\partial}{\partial t_2}\left[\omega^2\phi_{11}\frac{\partial\phi_{11}^*}{\partial z}+*\right] + \frac{1}{g}\frac{\partial}{\partial t_1}\Big[i\omega\frac{\partial\phi_{11}^*}{\partial z}\frac{\partial\phi_{11}}{\partial t_1} - i\omega\phi_{11}\frac{\partial^2\phi_{11}^*}{\partial z\,\partial t_1}\Big) + *\Big] \\
& \underline{+\frac{1}{g}\frac{\partial}{\partial t_1}\left[\left(\omega^2\phi_{11}\frac{\partial\phi_{21}^*}{\partial z}+\omega^2\phi_{21}\frac{\partial\phi_{11}^*}{\partial z}\right)+*\right]} - \frac{\partial}{\partial x_1}\left[\left(i\omega\phi_{11}^*\frac{\partial\phi_{11}}{\partial x_1} - ik\phi_{11}^*\frac{\partial\phi_{11}}{\partial t_1}\right)+*\right] \\
& \underline{-2\frac{\partial}{\partial x_1}(\omega k\phi_{21}\phi_{11}^*+*)} - \frac{\partial}{\partial x_1}\left(\frac{\partial\phi_{00}}{\partial x_1}\frac{\partial\phi_{10}}{\partial t_1}+\frac{\partial\phi_{00}}{\partial t_1}\frac{\partial\phi_{10}}{\partial x_1}\right) + 2\omega k\frac{\partial}{\partial x_2}|\phi_{11}|^2 \\
& -\frac{\partial}{\partial x_1}\left(\frac{\partial\phi_{00}}{\partial x_1}\frac{\partial\phi_{00}}{\partial t_2}\right) - \frac{\partial}{\partial y_1}\left(i\omega\phi_{11}^*\frac{\partial\phi_{11}}{\partial y_1}+*\right) - \frac{\partial}{\partial y_1}\left(\frac{\partial\phi_{00}}{\partial y_1}\frac{\partial\phi_{10}}{\partial t_1}+\frac{\partial\phi_{00}}{\partial t_1}\frac{\partial\phi_{10}}{\partial y_1}\right) \\
& -\frac{\partial}{\partial y_1}\left(\frac{\partial\phi_{00}}{\partial y_1}\frac{\partial\phi_{00}}{\partial t_2}\right) + \frac{1}{2g}\frac{\partial^2}{\partial z\,\partial t_1}\Big\{2\frac{\partial\phi_{00}}{\partial t_1}\left(k^2|\phi_{11}|^2+\left|\frac{\partial\phi_{11}}{\partial z}\right|^2\right) \\
& +4\frac{\partial\phi_{00}}{\partial x_1}(-k\omega|\phi_{11}|^2)\Big\} - \frac{2}{g^2}\frac{\partial}{\partial t_1}\left[\frac{\partial\phi_{00}}{\partial t_1}\left(\omega^2\left|\frac{\partial\phi_{11}}{\partial z}\right|^2\right)\right] - \frac{1}{g^2}\frac{\partial}{\partial t_1}\left[\frac{\partial\phi_{00}}{\partial t_1}\left(\omega^2\phi_{11}\frac{\partial^2\phi_{11}^*}{\partial z^2}+*\right)\right] \\
& -\frac{1}{2}\frac{\partial}{\partial x_1}\left[\frac{\partial\phi_{00}}{\partial x_1}\left(\left(\frac{\partial\phi_{00}}{\partial x_1}\right)^2+\left(\frac{\partial\phi_{00}}{\partial y_1}\right)^2\right)\right] - \frac{\partial}{\partial x_1}\Big[\frac{\partial\phi_{00}}{\partial x_1}\left(k^2|\phi_{11}|^2+\left|\frac{\partial\phi_{11}}{\partial z}\right|^2\right) \\
& +2\frac{\partial\phi_{00}}{\partial x_1}(k^2|\phi_{11}|^2)\Big] + \frac{1}{g}\frac{\partial}{\partial x_1}\left[\frac{\partial\phi_{00}}{\partial x_1}\left(\omega^2\phi_{11}^*\frac{\partial\phi_{11}}{\partial z}+*\right)+\frac{\partial\phi_{00}}{\partial t_1}\left(-k\omega\phi_{11}^*\frac{\partial\phi_{11}}{\partial z}+*\right)\right] \\
& +\frac{1}{g}\frac{\partial}{\partial x_1}\left[\frac{\partial\phi_{00}}{\partial t_1}\left(-k\omega\phi_{11}^*\frac{\partial\phi_{11}}{\partial z}+*\right)\right] - \frac{1}{2}\frac{\partial}{\partial y_1}\left[\frac{\partial\phi_{00}}{\partial y_1}\left[\left(\frac{\partial\phi_{00}}{\partial x_1}\right)^2+\left(\frac{\partial\phi_{00}}{\partial y_1}\right)^2\right]\right] \\
& -\frac{\partial}{\partial y_1}\left[\frac{\partial\phi_{00}}{\partial y_1}\left(k^2|\phi_{11}|^2+\left|\frac{\partial\phi_{11}}{\partial z}\right|^2\right)\right] + \frac{1}{g}\frac{\partial}{\partial y_1}\left[\frac{\partial\phi_{00}}{\partial y_1}\left(\omega^2\phi_{11}\frac{\partial\phi_{11}^*}{\partial z}+*\right)\right].
\end{aligned} \tag{C 1}
$$

Most of the terms above contribute to the secular term of (3.55). The secular terms involving B must come from terms containing ϕ_{21} as underlined in (C 1).

Appendix D. An estimate of bottom friction

In estimating bottom friction on the long trapped waves, we first ignore nonlinear effects and then add friction damping to nonlinear damping. From energy consideration of linear long-wave theory it is easily shown that

$$\frac{\partial}{\partial t}\int_{-\infty}^{0,\infty}(\tfrac{1}{2}u^2h+\tfrac{1}{2}g\zeta^2)\,dx = -\int_{-\infty}^{0,\infty}\tau_b u\,dx, \tag{D 1}$$

where the integral is over the entire fluid domain and τ_b the bottom stress. The left-hand side is simply $\partial\mathscr{E}/\partial t$, where $\mathscr{E}$ is the total energy in the wave. On the right-hand side, we introduce

$$\tau_b = \frac{4}{3\pi}\frac{\rho f U_0}{h_0}uh, \tag{D 2}$$

where U_0 and h_0 are constant characteristic velocity and depth respectively. Equation (D 2) may be regarded as an equivalent linearization of the usual formula

$$\tau_b = \tfrac{1}{2}\rho f|u|u, \tag{D 3}$$

where f is the empirical coefficient and U_0 is just the amplitude of u. Substituting (D 2) into (D 1) and invoking equipartition, we get

$$\frac{\partial \mathscr{E}}{\partial t} = -\frac{4}{3\pi}\frac{fU_0}{h_0}\mathscr{E}. \tag{D4}$$

Since $\mathscr{E} \propto |D|^2$ we have

$$\frac{\partial |D|}{\partial \bar{t}_3} = -\frac{2fU_0}{3\pi\omega\epsilon^3 h_0}|D|. \tag{D 5}$$

The dimensionless damping constant is then

$$\mu = \frac{2}{3\pi}\frac{fU_0}{\omega\epsilon^3 h^\circ}. \tag{D 6}$$

Estimating $\omega = 2\pi/10\ \mathrm{s}^{-1}$, $f = 0{\cdot}01$, $h^\circ = 10\,\mathrm{m}$, $\epsilon = 0{\cdot}05$ and $U_0 = 0{\cdot}1\ \mathrm{m\ s}^{-1}$, we get $\mu \sim 0{\cdot}3 = O(1)$. Since (D 4) resembles the linear part of (D 2) we simply change $\gamma_0 + \langle A^{\circ 2}\rangle\gamma_1$ to $\gamma_0 + \langle A^{\circ 2}\rangle\gamma_1 + \omega\mu$ in (4.2). In all the computations we have taken $\mu = 1$.

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